FM 431: Econometrics of Financial Markets

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- The slides were produced using LaTeX and R (the R project; www.R-project.org) on a Linux system.
- Course material is available at www.hs-stat.com/courses/FM431.

Chapter 3:

ARMA Models

(PART)

3.1 Introduction

ARMA models: scope and outlook.

- ARMA models are a class of stochastic processes.
- The idea behind ARMA models is:

> Exploit the autocorrelation structure of the series!

- In this chapter, we shall see some properties and the limitations of ARMA processes in financial modeling.

3.1 Introduction

The simplest ARMA models.

- MA(1): $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$
- AR(1): $X_t = c + aX_{t-1} + \epsilon_t$
- ARMA(1,1): $X_t = c + aX_{t-1} + \epsilon_t + \beta\epsilon_{t-1}$

(ϵ_t) is white noise.

3.2 MA Processes

MA: Definition.

An MA(q) process (X_t) is defined as

$$X_t \quad = \quad c + \epsilon_t + \beta_1 \epsilon_{t-1} + \ldots + \beta_q \epsilon_{t-q}$$

or, equivalently,

$$X_t \quad = \quad c + \beta(L)\epsilon_t,$$

where $\beta(L)$ is a polynomial of degree q in L.

- L is the lag operator.
- (ϵ_t) is white noise.

3.2 MA Processes

MA(1): Its autocorrelation function.

For an MA(1) process: $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$

$$\begin{aligned}
\text{var}(X_t) &= \text{var}(\epsilon_t + \beta\epsilon_{t-1}) = (1+\beta^2)\sigma_\epsilon^2, \\
\text{cov}(X_t, X_{t+1}) &= \text{cov}(\epsilon_t + \beta\epsilon_{t-1}, \epsilon_{t+1} + \beta\epsilon_t) = \beta\sigma_\epsilon^2, \\
\text{cov}(X_t, X_{t+s}) &= 0 \quad \text{for } s \geq 2.
\end{aligned}$$

The acf is therefore:

$$s \mapsto \rho(s) = \begin{cases} \beta/(1+\beta^2) & \text{for } s = 1, \\ 0 & \text{for } s \geq 2 \end{cases}$$

The acf of any MA process cuts off.

3.2 MA Processes

MA(1): Its unconditional moments.

For an MA(1) process $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$:

$$\begin{aligned} \mathsf{E}(X_t) &= c, \\ \mathsf{var}(X_t) &= (1+\beta^2)\sigma_\epsilon^2. \end{aligned}$$

3.2 MA Processes

“Inverting” an MA(1) process.

For an MA(1) process $X_t = \epsilon_t + \beta\epsilon_{t-1}$, we can write:

$$\begin{aligned}
X_t &= (1+\beta L)\epsilon_t, \\
\epsilon_t &= \frac{X_t}{1+\beta L}, \\
\epsilon_t &= \sum_{s=0}^{\infty}(-\beta)^s X_{t-s}, \\
X_t &= \epsilon_t + \beta\sum_{s=0}^{\infty}(-\beta)^s X_{t-s-1}.
\end{aligned}$$

3.2 MA Processes

MA(1): Its conditional moments.

Therefore:

$$\begin{aligned} \mathsf{E}(X_t | X_{t-1}, \ldots) &= \beta \sum_{s=0}^{\infty} (-\beta)^s X_{t-s-1}, \\ \mathsf{var}(X_t | X_{t-1}, \ldots) &= \sigma_\epsilon^2. \end{aligned}$$

We observe:

- An MA process can be forecast only if it is invertible.
- MA is a conditional expectation model.

3.3 AR Processes

AR: Definition. An AR(p) process (X_t) is defined as

$$X_t = c + a_1 X_{t-1} + \ldots + a_p X_{t-p} + \epsilon_t$$

or, equivalently,

$$a(L) X_t = c + \epsilon_t,$$

where $a(L)$ is a polynomial of degree p in L, such that (X_t) is stationary.

- L is the lag operator.
- (ϵ_t) is white noise.

3.3 AR Processes

AR(1): Its autocorrelation function.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$, we can write:

$$\begin{aligned}
(1 - aL)X_t &= c + \epsilon_t, \\
X_t &= \frac{c}{1 - aL} + \frac{\epsilon_t}{1 - aL}, \\
X_t &= \frac{c}{1 - a} + \sum_{s=0}^{\infty} a^s \epsilon_{t-s}.
\end{aligned}$$

This can be used to show that the acf of this process is

$$s \mapsto \rho(s) = a^s.$$

3.3 AR Processes

AR(1): Its unconditional moments.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$:

$$\begin{aligned} \mathsf{E}(X_t) &= \frac{c}{1-a}, \\ \mathsf{var}(X_t) &= \frac{\sigma_\epsilon^2}{1-a^2}. \end{aligned}$$

3.3 AR Processes

AR(1): Its conditional moments.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$:

$$\begin{aligned} \mathsf{E}(X_t | X_{t-1}, \ldots) &= c + aX_{t-1}, \\ \mathsf{var}(X_t | X_{t-1}, \ldots) &= \sigma_\epsilon^2. \end{aligned}$$

As for an MA process, we observe:

- AR is a conditional expectation model.

3.4 Mixed AR/MA Processes

ARMA processes.

A mixed model

$$a(L)X_t = c + \beta(L)\epsilon_t$$

where $a(L)$ is a polynomial of degree p in L and $\beta(L)$ is a polynomial of degree q in L is called an ARMA(p, q) process, provided that it is stationary.

3.4 Mixed AR/MA Processes

ARIMA processes.

Consider a process (X_t) such that

$$Z_t = (1 - L)X_t = X_t - X_{t-1}$$

is an ARMA(p, q) process.
Then, (X_t) is called an ARIMA$(p, 1, q)$ process.

Keywords:

- stochastic trend,
- differencing.

3.5 Seasonal ARMA Models

Consider the case of monthly data.

A simple multiplicative seasonal model is:

$$(1 - a_{12}L^{12})(1 - a_1 L)X_t = c + (1 - \beta_{12}L^{12})(1 - \beta_1 L)\epsilon_t$$

This is called a SARMA(1,1)×(1,1)$_{12}$ process.

Observe that there is, in particular, a direct impact of X_{t-13} on X_t.

3.6 Tentative Identification

Goals, aspects, tools of model identification.

- Identification means: Find a stochastic model (here: an AR(I)MA model) which may have created the observed series.
- Most important tools in model identification: acf / pacf
- The procedure is:
  - Determine the empirical acf / pacf.
  - Find an ARMA process with similar acf / pacf.

3.6 Tentative Identification

Goals, aspects, tools of model identification.

Considerations:

- The model should be simple ("parsimonious").
- The residuals should have no more autocorrelation structure. (They should be white noise. — What does this mean?)

3.6 Tentative Identification

Concerning model simplicity:

The Akaike information criterion (AIC). It is computed as:

$$\text{AIC} = T \cdot \ln(\text{residual sum of squares}) + 2n,$$

where:

n = number of parameters estimated (typically, $p + q + 1$),
T = number of usable observations.

- A model should be selected such that AIC becomes small.
- AIC penalizes the use of additional parameters.

3.6 Tentative Identification

Concerning the residuals:

The Box-Ljung statistic.

It permits to test the null hypothesis

H_0 : There is no autocorrelation in the residuals up to lag s.

against the alternative

H_1 : There IS autocorrelation in the residuals up to lag s.

3.6 Tentative Identification

Concerning the residuals:

The Box-Ljung statistic.

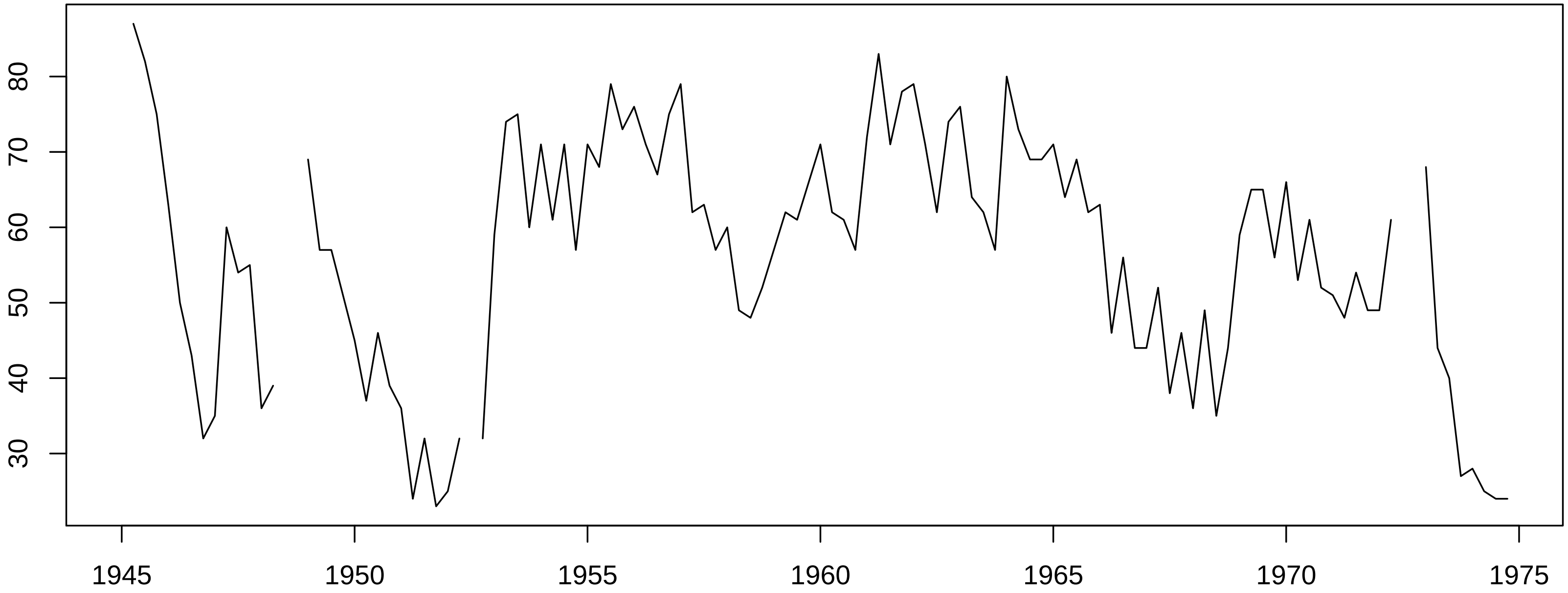
It is defined as

$$Q = T(T+2)\sum_{k=1}^{s}\frac{r_k^2}{T-k},$$

where r_k is the empirical autocorrelation of the residuals.
Critical: large values.

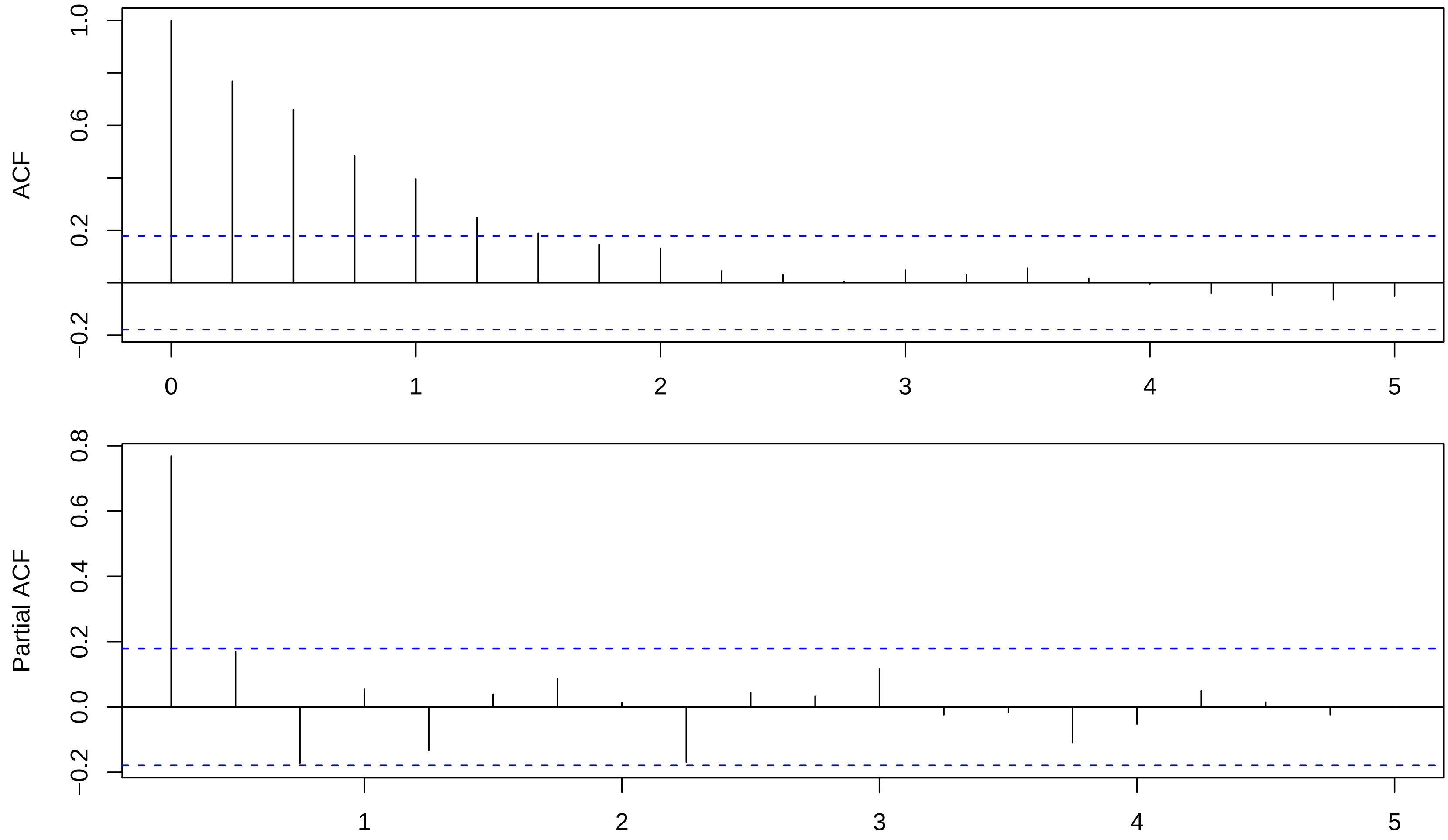
3.7 Example: Approval Rates

Quarterly Approval Ratings of US Presidents.



3.7 Example: Approval Rates

Acf, pacf of the series.

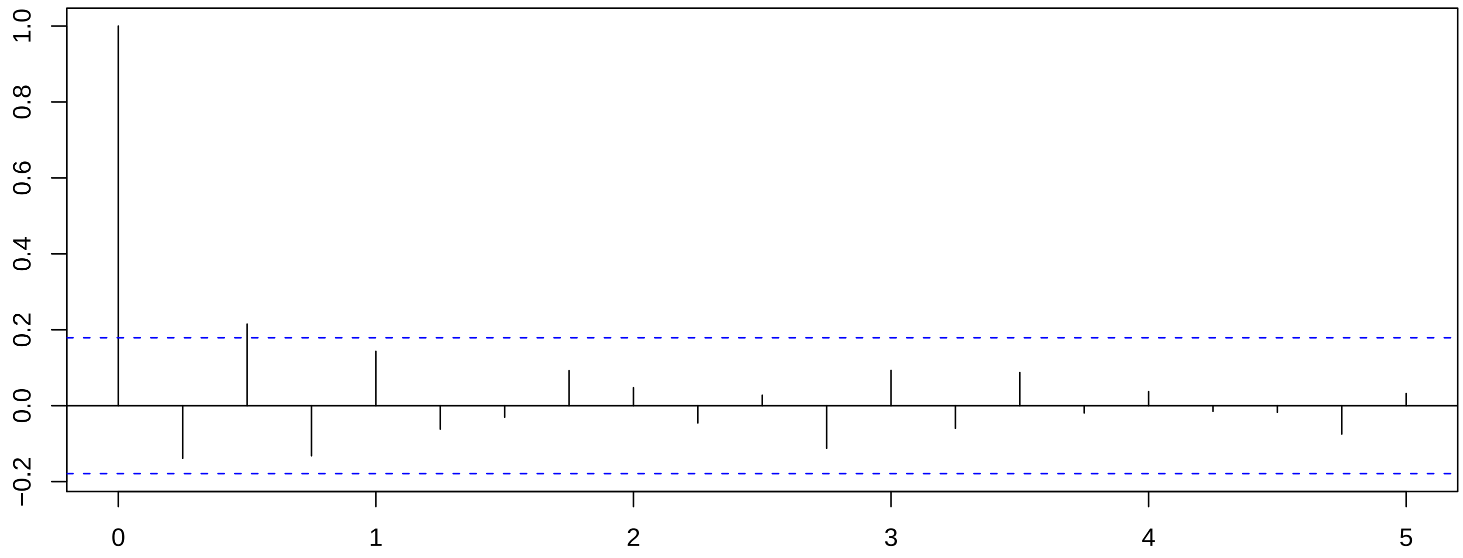


3.7 Example: Approval Rates

First trial: ARMA(1,0).

```
Coefficients:
         ar1  intercept
      0.8242    56.1505
s.e.  0.0555     4.6434

sigma^2 estimated as 85.47;  log likelihood = -416.89;  aic = 839.78
```

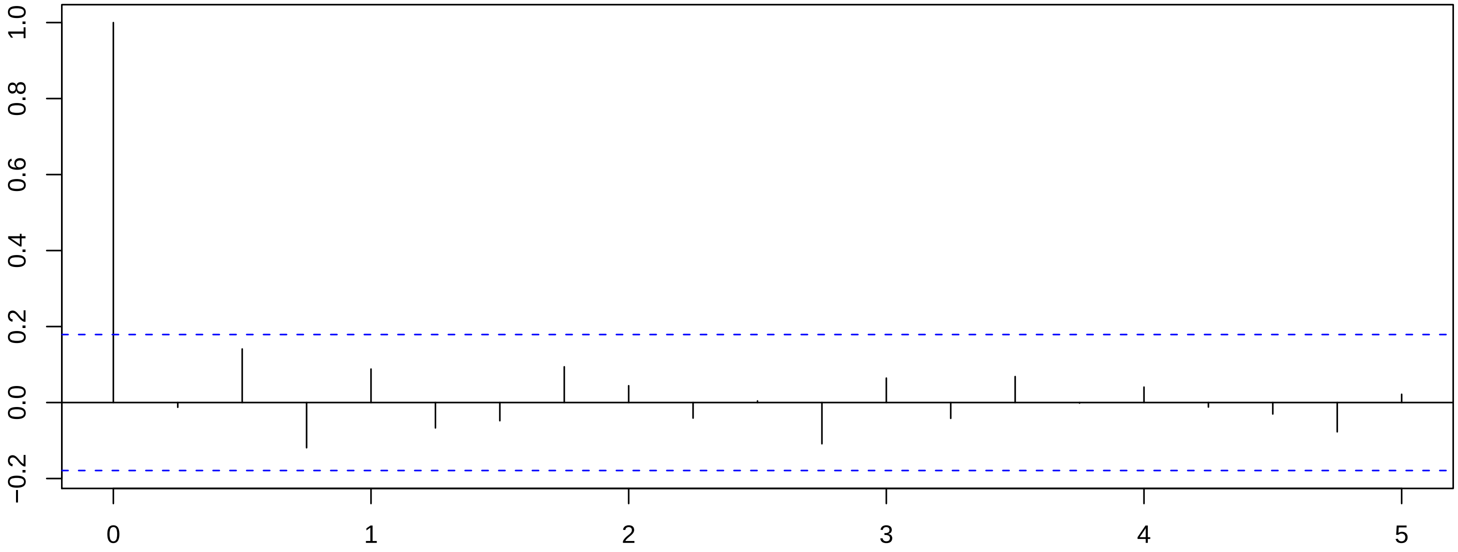


3.7 Example: Approval Rates

Another trial: ARMA(2,0).

```
Coefficients:
         ar1     ar2  intercept
      0.7187  0.1339    56.0554
s.e.  0.0969  0.1010     5.4184

sigma^2 estimated as 84.32;  log likelihood = -416.02;  aic = 840.05
```

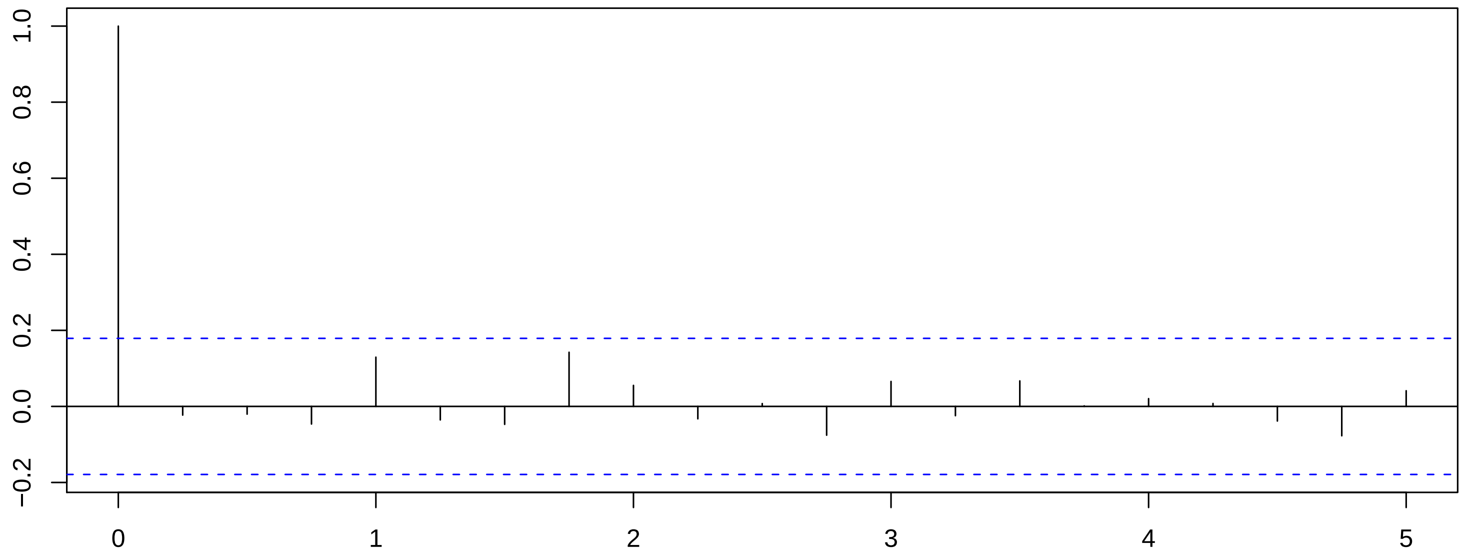


3.7 Example: Approval Rates

Final trial: ARMA(3,0).

```
Coefficients:
         ar1     ar2      ar3  intercept
      0.7496  0.2523  -0.1890    56.2223
s.e.  0.0936  0.1140   0.0946     4.2845

sigma^2 estimated as 81.12;  log likelihood = -414.08;  aic = 838.16
```



3.7 Example: Approval Rates

Final trial: ARMA(3,0).