FEC 522:
Financial Econometrics II

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- R files used for this course are available upon request.

Chapter 6:

VAR Models

6.1 VAR: Alternative Model Specifications

Structural (= "primitive") form (lag 1, omitting the constants).

$$\begin{aligned} x_{1t} &= -b_{12}x_{2t} + \gamma_{11}x_{1t-1} + \gamma_{12}x_{2t-1} + \epsilon_{1t} \\ x_{2t} &= -b_{21}x_{1t} + \gamma_{21}x_{1t-1} + \gamma_{22}x_{2t-1} + \epsilon_{2t} \end{aligned}$$

Equivalently:

$$\mathtt{B} \cdot x_t = \Gamma_1 \cdot x_{t-1} + \epsilon_t$$

with

$$x_t = \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}, \quad \mathtt{B} = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix},$$

(ϵ_t): bivariate white noise, uncorrelated

6.1 VAR: Alternative Model Specifications

Questions etc.

- Questions:
  - interpretation of the model?
  - contemporaneous impact?
  - estimation of parameters?
- Purpose:
  - investigation of interaction
  - forecasting
  - assessment of spillovers

6.1 VAR: Alternative Model Specifications

Standard form.

Re-writing the model:

$$x_t = \mathtt{A}_1 x_{t-1} + e_t$$

with

$$\mathtt{A}_1 = \mathtt{B}^{-1} \cdot \Gamma_1, \quad e_t = \mathtt{B}^{-1} \cdot \epsilon_t$$

- this is called the "standard form"
- seemingly, no more contemporaneous impact
- errors are correlated
- estimation, identification?

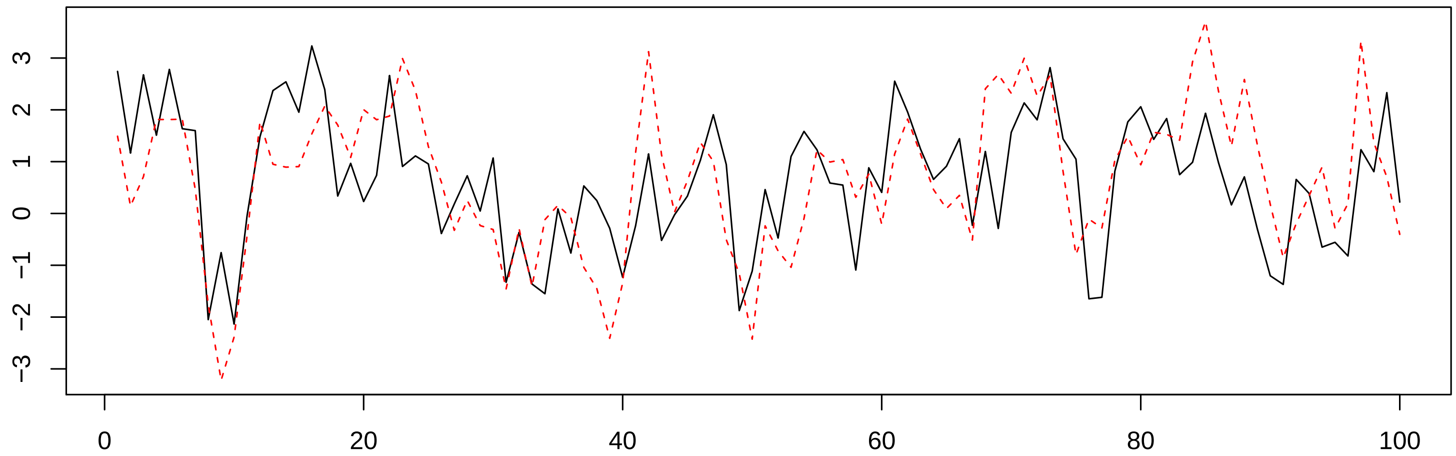
6.2 Estimation and Identification

A simulation example.

- Parameters:

$$\mathtt{B} = \begin{pmatrix} 1 & -0.8 \\ 0.2 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0.54 & -0.36 \\ 0.20 & 0.70 \end{pmatrix},$$

- Plot (part only):



6.2 Estimation and Identification

A simulation example.

Output when regressing x_{1t} on x_{2t}, x_{1t-1}, x_{2t-1}:

```
Call:
lm(formula = x1 ~ 0 + x2 + x1.lag1 + x2.lag1)

Residuals:
     Min       1Q   Median       3Q      Max
-2.85820 -0.64004  0.00679  0.76603  2.80106

Coefficients:
        Estimate Std. Error t value Pr(>|t|)
x2       0.55568    0.03691   15.06  < 2e-16 ***
x1.lag1  0.54674    0.02608   20.96  < 2e-16 ***
x2.lag1 -0.15130    0.04123   -3.67 0.000255 ***

Residual standard error: 1.006 on 997 degrees of freedom
Multiple R-squared: 0.5991,Adjusted R-squared: 0.5979
F-statistic: 496.6 on 3 and 997 DF,  p-value: < 2.2e-16
correlation, x2 and eps1: -0.1072276
```

6.2 Estimation and Identification

The problems.

- VAR in structural form: cannot be estimated.
- Idea:
  estimate standard form first, then find structural form from it.
- However: # parameters. . .
- Idea: restrict the primitive system. . .

6.2 Estimation and Identification

Structural form with $b_{21} = 0$ (Sims, 1980).

$$\begin{aligned} x_{1t} &= -b_{12} x_{2t} + \gamma_{11} x_{1t-1} + \gamma_{12} x_{2t-1} + \epsilon_{1t} \\ x_{2t} &= \gamma_{21} x_{1t-1} + \gamma_{22} x_{2t-1} + \epsilon_{2t} \end{aligned}$$

Hence:

$$\mathtt{B} = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad \mathtt{B}^{-1} = \begin{pmatrix} 1 & -b_{12} \\ 0 & 1 \end{pmatrix}$$

The covariance matrix of e_t is:

$$\begin{pmatrix} \sigma_1^2 + b_{12}\sigma_2^2 & -b_{12}\sigma_2^2 \\ -b_{12}\sigma_2^2 & \sigma_2^2 \end{pmatrix}$$

6.2 Estimation and Identification

Remarks about the system when $b_{21} = 0$.

- The system is recursive.
- x_{1t} has no contemporaneous effect on x_{2t}.
- x_{2t} is called "causally prior" to x_{1t}.
- Both ϵ_{1t} and ϵ_{2t} affect the contemporaneous value of x_{1t}.
- Only ϵ_{2t} affects the contemporaneous value of x_{2t}.
- Observed e_{2t} are completely attributed to shocks to x_{2t}.

6.3 Impulse Response Functions

The case of AR(1).

- AR(1):

$$x_t = c + a\, x_{t-1} + \epsilon_t$$

- AR(1) as MA(∞):

$$x_t = \mu + \sum_{i=0}^{\infty} a^i\, \epsilon_{t-i} = \sum_{i=0}^{\infty} \Phi(i)\, \epsilon_{t-i}$$

- What is the effect of a one-unit shock in ϵ_t on $x_t,\ x_{t+1},\ x_{t+2},\ \ldots$?

6.3 Impulse Response Functions

The case of a VAR.

- Standard form:

$$x_t = \mathtt{A}_0 + \mathtt{A}_1\, x_{t-1} + e_t$$

- In MA(∞) representation:

$$x_t = \mu + \sum_{i=0}^{\infty} \mathtt{A}_1^i\, e_{t-i} = \mu + \sum_{i=0}^{\infty} \mathtt{A}_1^i\, \mathtt{B}^{-1}\, \epsilon_{t-i} = \mu + \sum_{i=0}^{\infty} \Phi(i)\, \epsilon_{t-i}$$

- Here, $\Phi(i)$ is a matrix:

$$\Phi(i) = \mathtt{A}_1^i \cdot \mathtt{B}^{-1} = \begin{pmatrix} \Phi_{11}(i) & \Phi_{12}(i) \\ \Phi_{21}(i) & \Phi_{22}(i) \end{pmatrix}$$

- Effect of a one-unit shock in ϵ_t on x_t, x_{t+1}, x_{t+2}, . . . ?

6.3 Impulse Response Functions

Impulse response functions.

- Each $i \mapsto \Phi_{jk}(i)$ is called an impulse response function.
- "impact multipliers":

$$\Phi_{jk}(0)$$

- effect of a one-unit change in ϵ_{kt} on x_{jt+1}:

$$\Phi_{jk}(1)$$

- "long-run impact multipliers", quantifying accumulated effects:

$$\sum_{i=0}^{\infty} \Phi_{jk}(i)$$

6.3 Impulse Response Functions

Example.

- Given:

$$\mathtt{A}_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}, \quad \sigma_2^2 = \text{var}(e_{1t}) = \text{var}(e_{2t}), \quad \text{cor}(e_{1t}, e_{2t}) = 0.8$$

- Assumption: $b_{21} = 0$
- Then:

$$\mathtt{B} = \begin{pmatrix} 1 & -0.8 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = \mathtt{B} \cdot \mathtt{A}_1 = \begin{pmatrix} 0.54 & -0.36 \\ 0.20 & 0.70 \end{pmatrix}$$

- Decomposed errors:

$$\begin{aligned} e_{1t} &= \epsilon_{1t} + 0.8\epsilon_{2t}, \\ e_{2t} &= \epsilon_{2t} \end{aligned}$$

6.3 Impulse Response Functions

Example.

- The model is:

$$
\begin{aligned}
x_{1t} &= 0.8x_{2t} + 0.54x_{1t-1} - 0.36x_{2t-1} + \epsilon_{1t} \\
x_{2t} &= 0.20x_{1t-1} + 0.70x_{2t-1} + \epsilon_{2t}
\end{aligned}
$$

or, equivalently:

$$
\begin{aligned}
x_{1t} &= 0.70x_{1t-1} + 0.20x_{2t-1} + e_{1t} \\
x_{2t} &= 0.20x_{1t-1} + 0.70x_{2t-1} + e_{2t}
\end{aligned}
$$

- What is the effect of a shock in ϵ_{1t} or ϵ_{2t} on x_t, x_{t+1}, ... ?

6.3 Impulse Response Functions

Example.

- time t (effect of a one-unit shock in ϵ_{2t} to x_{1t} and x_{2t}):
  - x_{2t} will jump by 1 unit
  - x_{1t} will jump by 0.8 units
- In terms of the impulse response function:

$$\Phi(0) = \mathtt{B}^{-1} = \begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix}$$

6.3 Impulse Response Functions

Example.

- time $t+1$ (effect of a one-unit shock in ϵ_{2t} to x_{1t+1} and x_{2t+1}):

$$\begin{array}{rcccccl} x_{2t} & = & 0.20x_{1t-1} & + & 0.70x_{2t-1} & + & e_{2t} \\ \text{will jump by:} & & 0.20 \cdot 0.80 & + & 0.70 \cdot 1 & & = 0.86 \end{array}$$

$$\begin{array}{rcccccl} x_{1t} & = & 0.70x_{1t-1} & + & 0.20x_{2t-1} & + & e_{1t} \\ \text{will jump by:} & & 0.70 \cdot 0.80 & + & 0.20 \cdot 1 & & = 0.76 \end{array}$$

- In terms of the impulse response function:

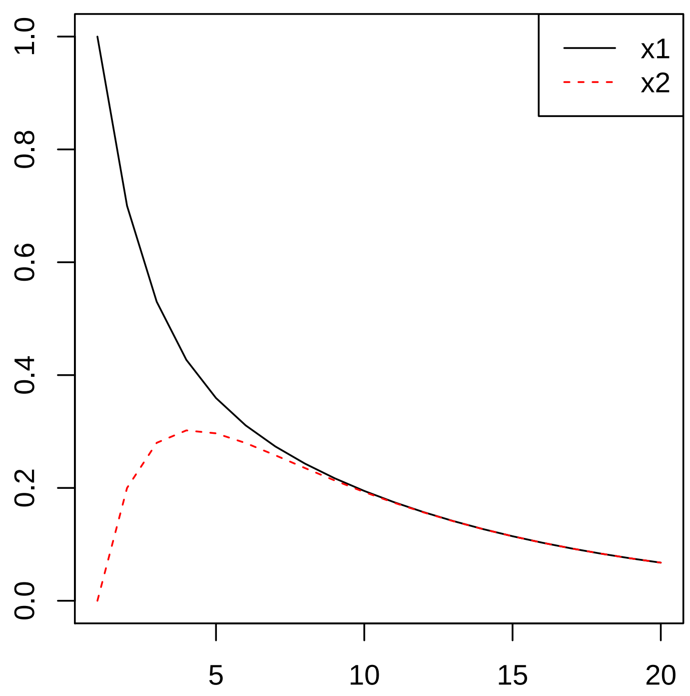
$$\Phi(1) = \mathtt{A}_1 \cdot \mathtt{B}^{-1} = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.70 & 0.76 \\ 0.20 & 0.86 \end{pmatrix}$$

6.3 Impulse Response Functions

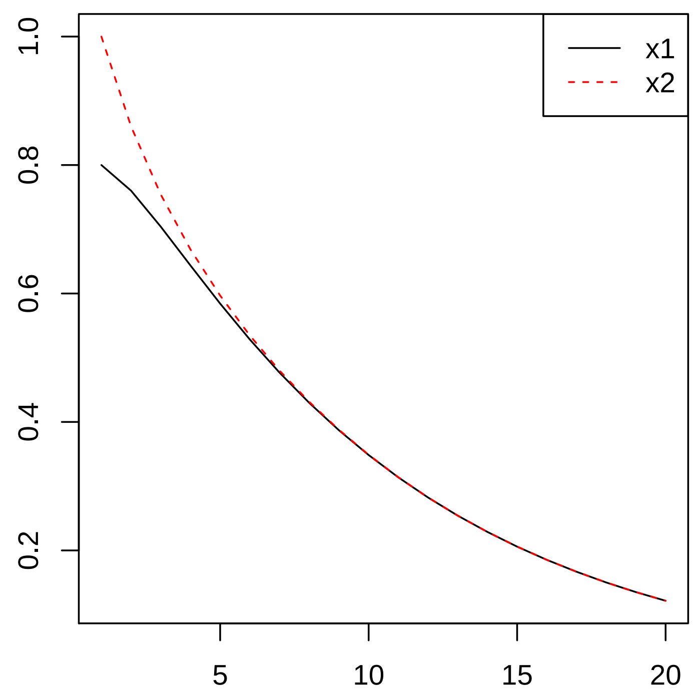
Example.

- impulse reponse functions:

of shock in ϵ_{1t}



of shock in ϵ_{2t}



6.3 Impulse Response Functions

Decompositions / orderings.

- Example: We assumed $b_{21} = 0$.
  What if we had assumed $b_{12} = 0$?

- A decomposition

$$\begin{aligned} e_{1t} &= \epsilon_{1t} - b_{12}\epsilon_{2t}, \\ e_{2t} &= \epsilon_{2t} \end{aligned}$$

  is called a ordering of the variables. (x_{2t}, causally prior)

6.3 Impulse Response Functions

Aspects when deciding which ordering might be appropriate.

- A priori knowledge of contemporaneous effects.
- Importance of ordering depends on the magnitude of $|\text{cor}(e_{1t}, e_{2t})|$.
  - $|\text{cor}(e_{1t}, e_{2t})| = 0$: ordering is immaterial.
  - $|\text{cor}(e_{1t}, e_{2t})| = 1$: single shock affects both variables contemporaneously.
  - Rule of thumb: If $|\text{cor}(e_{1t}, e_{2t})| > 0.2$ with 100 observations, there is significant correlation.
    Try different orderings, check for plausibility.
- . . . or use other means to identify the structural model!

6.4 Forecast Error Variance Decomposition

Forecasts and forecast errors.

- Given $\mathtt{A}_0$, $\mathtt{A}_1$, x_t, forecast x_{t+i}!

$i = 1$:

forecast	$\hat{x}_{t+1} = \mathtt{A}_0 + \mathtt{A}_1 x_t$
forecast error	e_{t+1}

$i = 2$:

forecast	$\hat{x}_{t+2} = (1 + \mathtt{A}_1)\ \mathtt{A}_0 + \mathtt{A}_1^2 x_t$
forecast error	$e_{t+2} + \mathtt{A}_1 e_{t+1}$

$i = n$:

forecast	$\hat{x}_{t+n} = \left(1 + \mathtt{A}_1 + \ldots + \mathtt{A}_1^{n+1}\right)\ \mathtt{A}_0 + \mathtt{A}_1^n x_t$
forecast error	$e_{t+n} + \mathtt{A}_1 e_{t+n-1} + \mathtt{A}_1^2 e_{t+n-2} + \ldots + \mathtt{A}_1^{n-1} e_{t+1}$

6.4 Forecast Error Variance Decomposition

The forecast error in terms of the structural model.

$$\sum_{i=0}^{n-1} \mathtt{A}_1^i \, e_{t+n-i} = \sum_{i=0}^{n-1} \mathtt{A}_1^i \, \mathtt{B}^{-1} \, \epsilon_{t+n-i} = \sum_{i=0}^{n-1} \Phi(i) \, \epsilon_{t+n-i}$$

For $x_{1\,t+n}$, the forecast error will be:

$$\sum_{i=0}^{n-1} \left(\Phi_{11}(i) \quad \Phi_{12}(i) \right) \begin{pmatrix} \epsilon_{1\,t+n-i} \\ \epsilon_{2\,t+n-i} \end{pmatrix}$$

$$\begin{array}{ccccccc} = & \Phi_{11}(0) & \epsilon_{1\,t+n} & + & \Phi_{12}(0) & \epsilon_{2\,t+n} & + \\ & \vdots & & & \vdots & & \\ & \Phi_{11}(n-1) & \epsilon_{1\,t+1} & + & \Phi_{12}(n-1) & \epsilon_{2\,t+1} & \end{array}$$

6.4 Forecast Error Variance Decomposition

Decomposition of the forecast error variance.

For $x_{1\,t+n}$, the forecast error variance is:

$$\text{var}\left(\sum_{i=0}^{n-1} \begin{pmatrix}\Phi_{11}(i) & \Phi_{12}(i)\end{pmatrix} \begin{pmatrix} \epsilon_{1\,t+n-i} \\ \epsilon_{2\,t+n-i} \end{pmatrix}\right)$$

$$\begin{aligned} = \quad & \Phi_{11}^2(0) \quad \sigma_1^2 \quad + \quad \Phi_{12}^2(0) \quad \sigma_2^2 \quad + \\ & \quad \vdots \qquad\qquad\qquad\qquad \vdots \\ & \Phi_{11}^2(n-1) \quad \sigma_1^2 \quad + \quad \Phi_{12}^2(n-1) \quad \sigma_2^2 \end{aligned}$$

$$= \underbrace{\sigma_1^2 \cdot \sum_{i=0}^{n-1} \Phi_{11}^2(i)} \quad + \quad \underbrace{\sigma_2^2 \cdot \sum_{i=0}^{n-1} \Phi_{12}^2(i)}$$

due to. . . shocks in ϵ_{1t} shocks in ϵ_{2t}

6.4 Forecast Error Variance Decomposition

Spillover table and spillover index.

- Schematically:

		from. . .	
		ϵ_{1t}	ϵ_{2t}
to. . .	$x_{1\,t+n}$	$\square$	$\blacksquare$
	$x_{1\,t+n}$	$\blacksquare$	$\square$

- Each row of the spillover table defines a fevd.

- spillover index $= \dfrac{\sum \blacksquare}{\sum \blacksquare + \sum \square}$ (Diebold & Yilmaz, 2009)

6.4 Forecast Error Variance Decomposition

Example.

For $x_{1\,t+n}$, the forecast error variance decomposition (fevd) is:

	from. . .		from. . .	
n	ϵ_{1t}	ϵ_{2t}	ϵ_{1t}	ϵ_{2t}
	(absolute)		(relative)	
1	1.000	0.800	0.556	0.444
2	1.281	1.296	0.497	0.503
3	1.463	1.710	0.461	0.539
4	1.592	2.051	0.437	0.563
5	1.689	2.330	0.420	0.580
⋮	⋮	⋮	⋮	⋮
∞	2.068	3.531	0.369	0.631