FEC 522:
Financial Econometrics II

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- R files used for this course are available upon request.

Chapter 5:

GARCH Processes

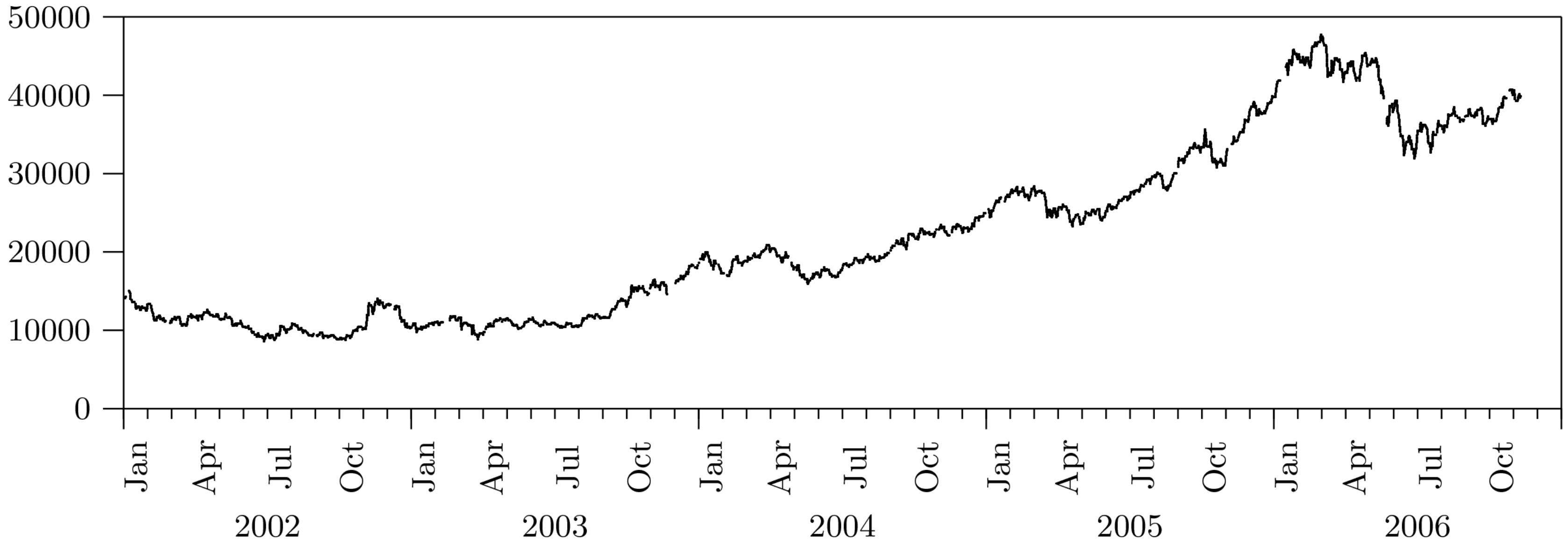
5.1 Introduction

GARCH models: scope and outlook.

- There is usually a certain form of heteroskedasticity in a series of returns.
- High volatility today can lead to high volatility tomorrow.
- Variances today and tomorrow are somehow related.
- For a time series (X_t) which is centered at 0, it holds that $\mathrm{var}(X_t) = \mathsf{E}(X_t^2)$.
- This form of heteroskedasticity implies that there will be autocorrelation in squared returns.

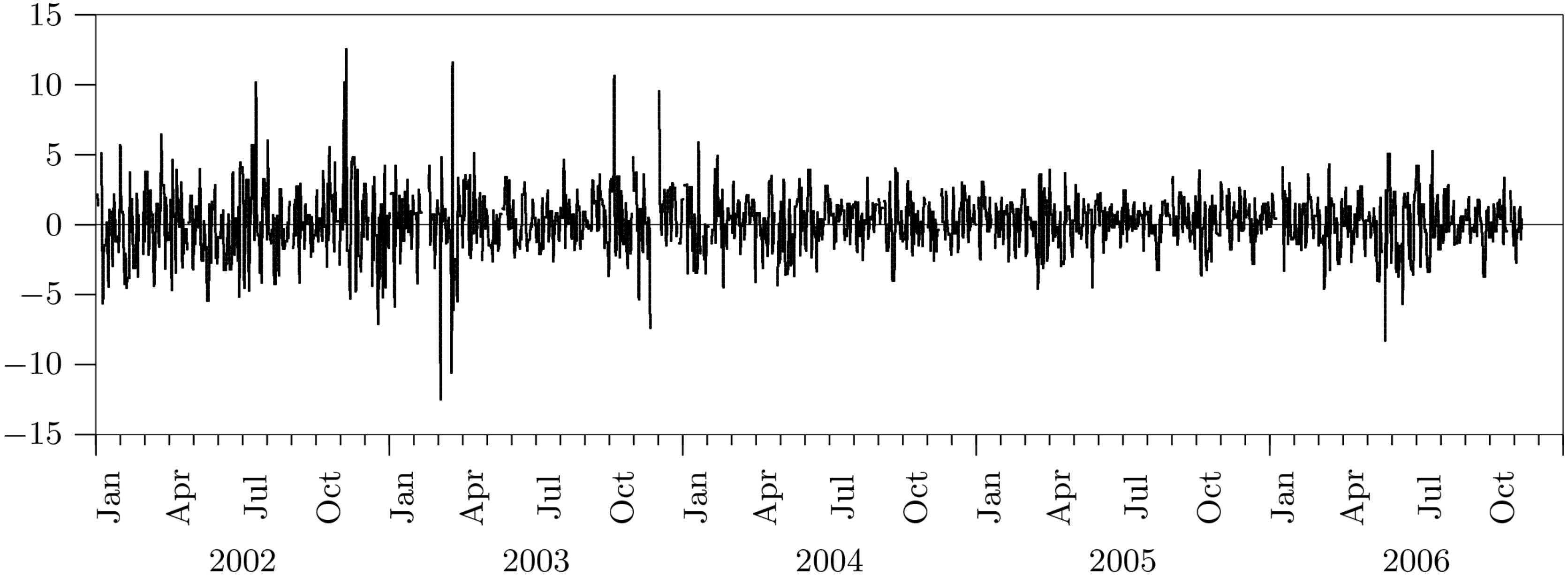
5.1 Introduction

Example: İMKB 100 — the level series.



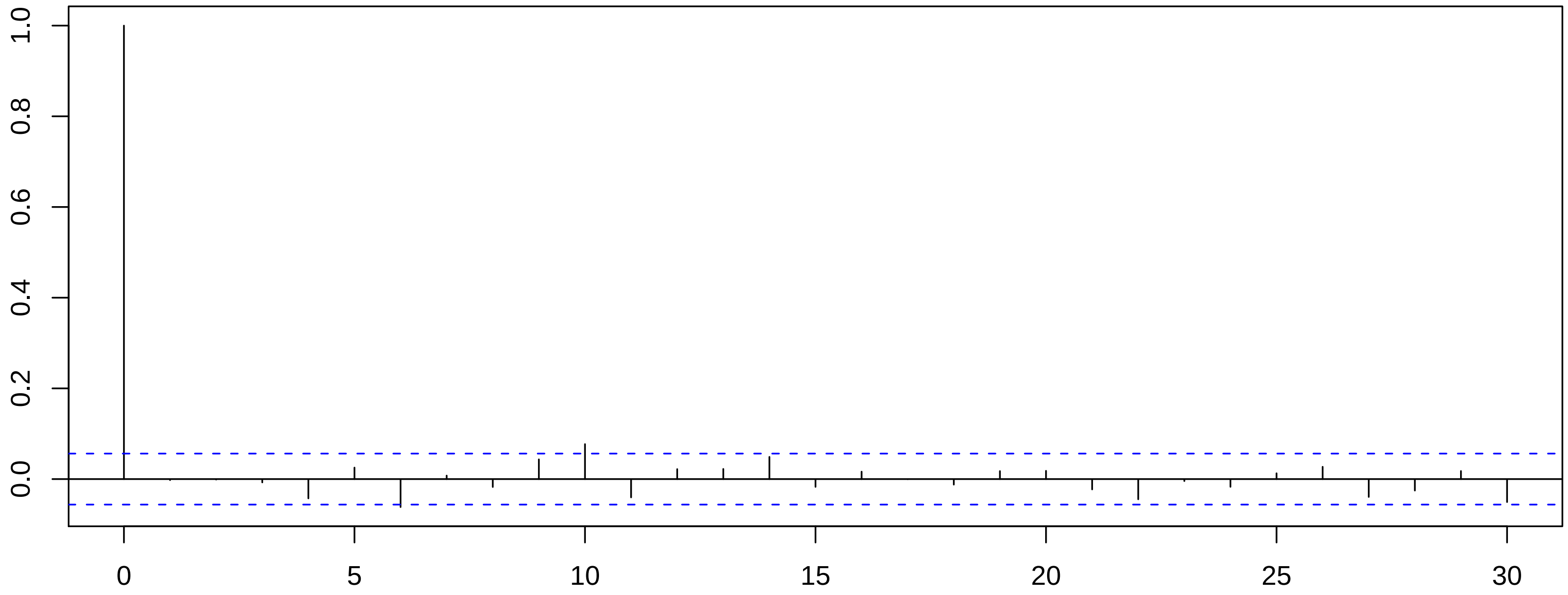
5.1 Introduction

Example: İMKB 100 — the return series.



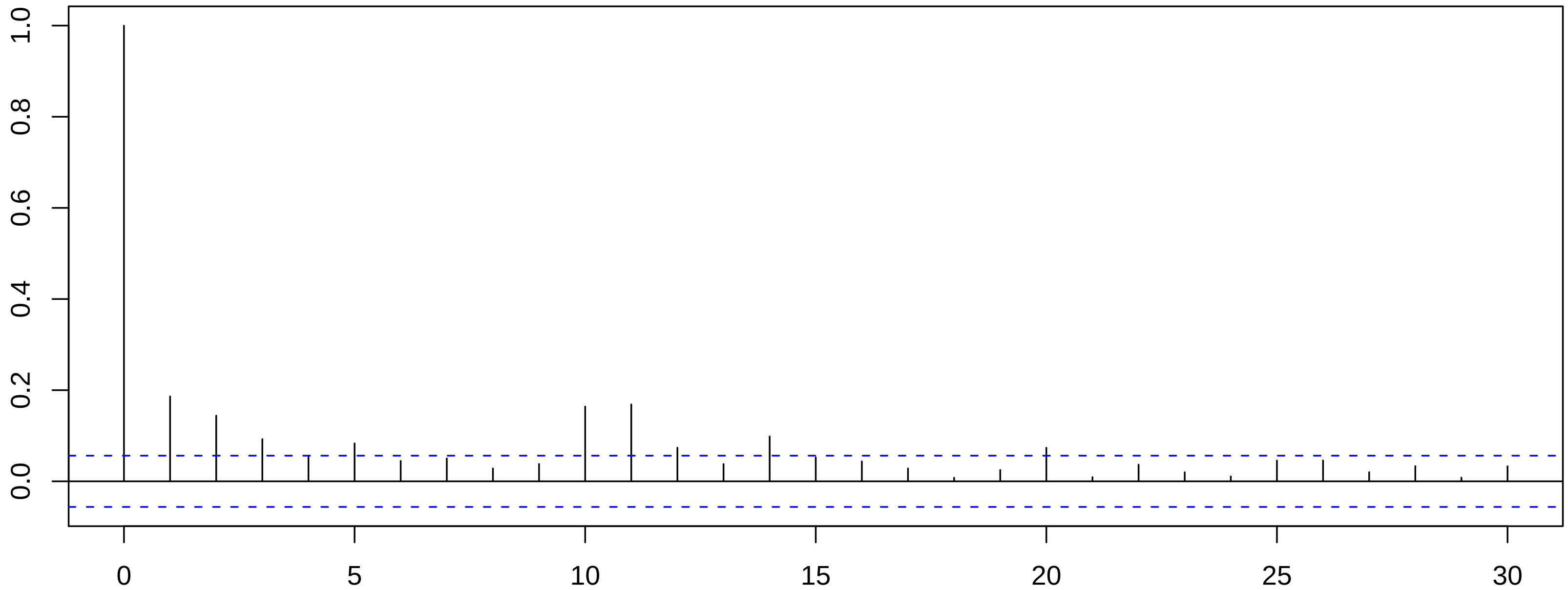
5.1 Introduction

Example: İMKB 100 — acf of the return series.



5.1 Introduction

Example: İMKB 100 — acf of the squared return series.



5.1 Introduction

GARCH models: scope and outlook.

- GARCH models are a class of stochastic processes which allow for this kind of heteroskedasticity.

- As before, the philosophy is:

  Find a stochastic model which may have created the observed series.

5.2 Definition

GARCH: Definition.

A GARCH(p, q) process (ϵ_t) is defined as

$$\epsilon_t \;=\; \nu_t \cdot \sqrt{h_t}, \quad (\nu_t)\text{: white noise with } \sigma_\nu^2 = \text{var}(\nu_t) = 1,$$

and

$$h_t \;=\; \alpha_0 + \underbrace{\sum_{i=1}^{q} \alpha_i \epsilon_{t-i}^2}_{\text{ARCH term}} + \underbrace{\sum_{i=1}^{p} \beta_i h_{t-i}}_{\text{GARCH term}}$$

with parameters $\alpha_0, \alpha_1, \ldots, \alpha_q, \beta_1, \ldots, \beta_p \geq 0$
and $\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i < 1$.

5.2 Definition

GARCH: Some first comments.

- The GARCH is a model for heteroskedastic time series.
- GARCH stands for *generalized autoregressive conditional heteroskedasticity*.
- The GARCH model was first developed by Engle (1982) and Bollerslev (1986).
- A GARCH process is stationary. (Heteroskedasticity need not be a reason for non-stationarity.)

5.2 Definition

GARCH: Some first comments. — GARCH$(1,1)$:

$$\begin{aligned} \epsilon_t &= \nu_t \cdot \sqrt{h_t}, \quad (\nu_t)\text{: white noise with } \sigma_\nu^2 = \text{var}(\nu_t) = 1, \\ h_t &= \alpha_0 + \underbrace{\alpha_1 \epsilon_{t-1}^2}_{\text{ARCH term}} + \underbrace{\beta_1 h_{t-1}}_{\text{GARCH term}} \end{aligned}$$

with parameters $\alpha_0, \alpha_1, \beta_1 \geq 0$ and $\alpha_1 + \beta_1 < 1$.

- Heteroskedasticity comes from the h_t term.
- The term h_t represents the one-period ahead forecast of the variance.
- The term ν_t represents “news”, “shocks”, “residuals”.

5.3 Conditional Heteroskedasticity

Conditional and unconditional moments of ARCH(1).

- Let's now consider an ARCH(1) process:

$$\epsilon_t = \nu_t \cdot \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$$

  with $\text{var}(\nu_t) = 1$, $\alpha_0 > 0$, $0 < \alpha_1 < 1$.

- Our program:
  - look at the unconditional moments,
  - look at the conditional moments.

5.3 Conditional Heteroskedasticity

Unconditional moments of ARCH(1).

- The model is:

$$\epsilon_t = \nu_t \cdot \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$$

- Unconditional expectation:

$$E(\epsilon_t) = E(\nu_t) \cdot E\left(\sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = 0$$

5.3 Conditional Heteroskedasticity

Unconditional moments of ARCH(1).

- The model is:

$$\epsilon_t = \nu_t \cdot \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$$

- Unconditional variance:

$$\text{var}(\epsilon_t) = E(\epsilon_t^2) = E(\nu_t^2) \cdot E\left(\alpha_0 + \alpha_1 \epsilon_{t-1}^2\right) = \frac{\alpha_0}{1 - \alpha_1}$$

- Unconditional correlation:

$$E(\epsilon_t \cdot \epsilon_{t+s}) = 0, \quad \text{so that} \quad \text{cor}(\epsilon_t, \epsilon_{t+s}) = 0 \quad \text{for } s \geq 1.$$

5.3 Conditional Heteroskedasticity

Conditional moments of ARCH(1).

- The model is:

$$\epsilon_t = \nu_t \cdot \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$$

- Conditional expectation:

$$E(\epsilon_t | \epsilon_{t-1}, \ldots) = E(\nu_t) \cdot E\left(\sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = 0$$

5.3 Conditional Heteroskedasticity

Conditional moments of ARCH(1).

- The model is:

$$\epsilon_t = \nu_t \cdot \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$$

- Conditional variance:

$$\text{var}(\epsilon_t | \epsilon_{t-1}, \ldots) = E(\epsilon_t^2 | \epsilon_{t-1}) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 = h_t$$

- This depends on ϵ_{t-1}! The ARCH is therefore a conditional variance model.

- There will be positive autocorrelation in the process (ϵ_t^2).

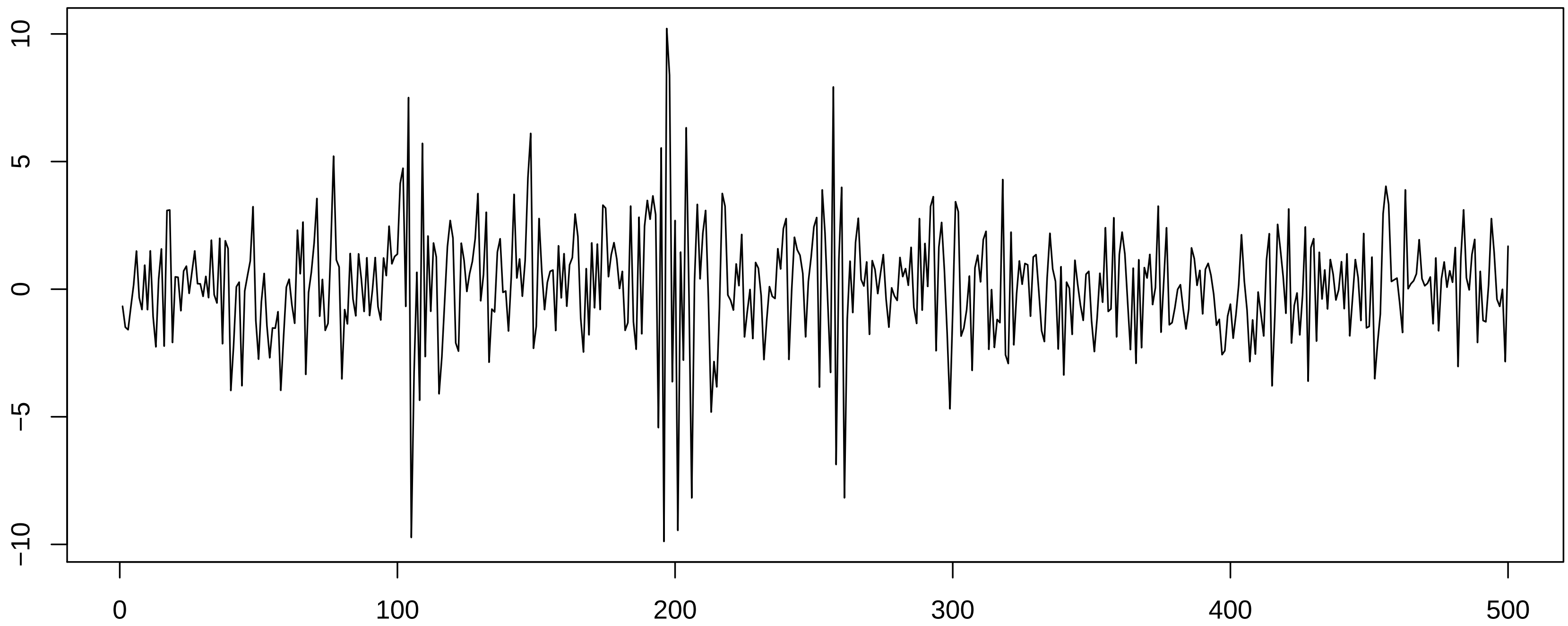
5.4 Simulation of a GARCH Process

The R code for simulating a GARCH(1,1) process.

```
a0 = 0.5;  a1 = 0.3;  b1 = 0.65

nu = rnorm(550)
epsi = rep(0, 550)
h = rep(0, 550)

for (i in 2:550) {
  h[i] = a0 + a1 * epsi[i-1]^2 + b1 * h[i-1]
  epsi[i] = nu[i] * sqrt(h[i])
}

epsi = epsi[51:550]
```

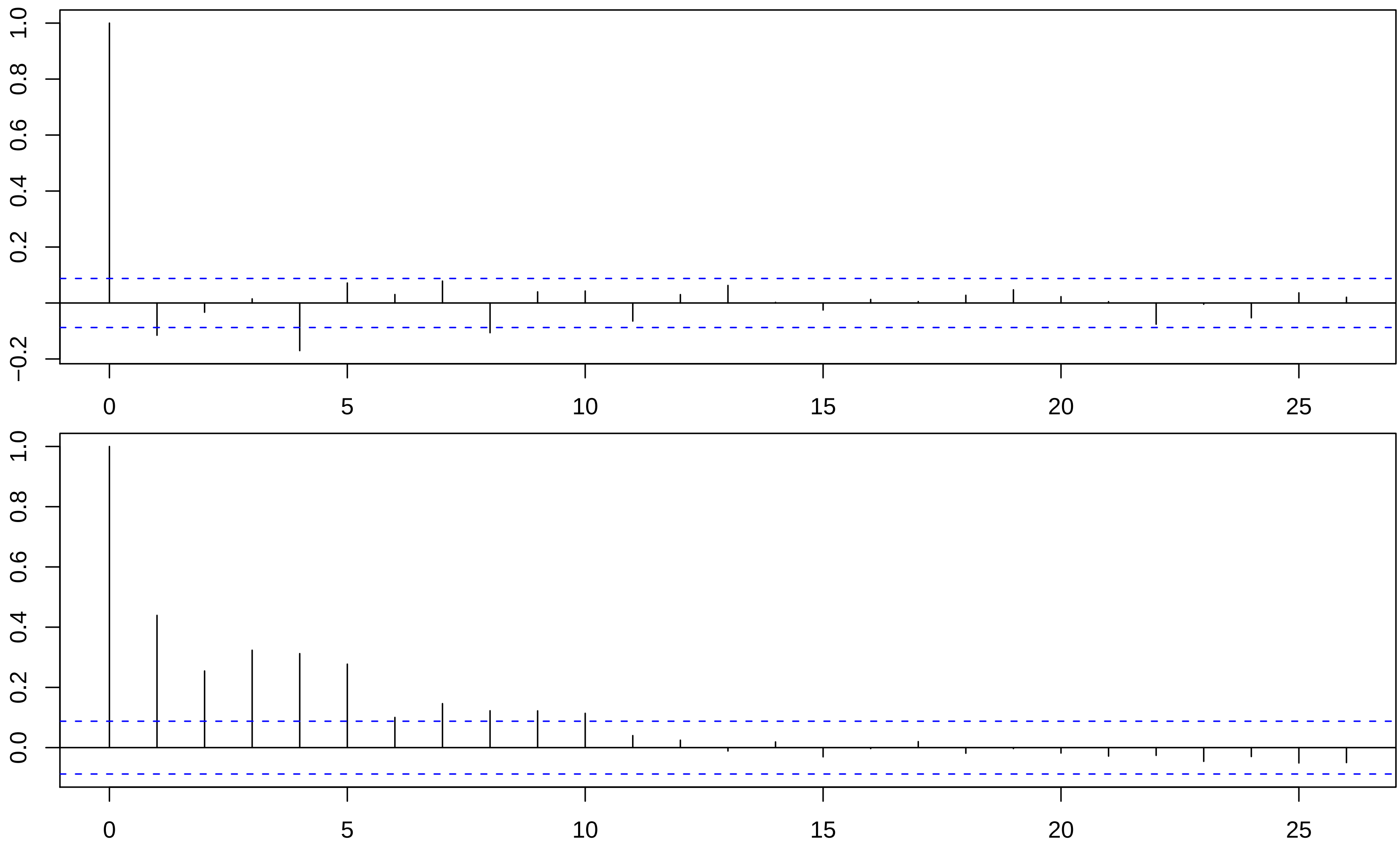
5.4 Simulation of a GARCH Process

A simulated series.



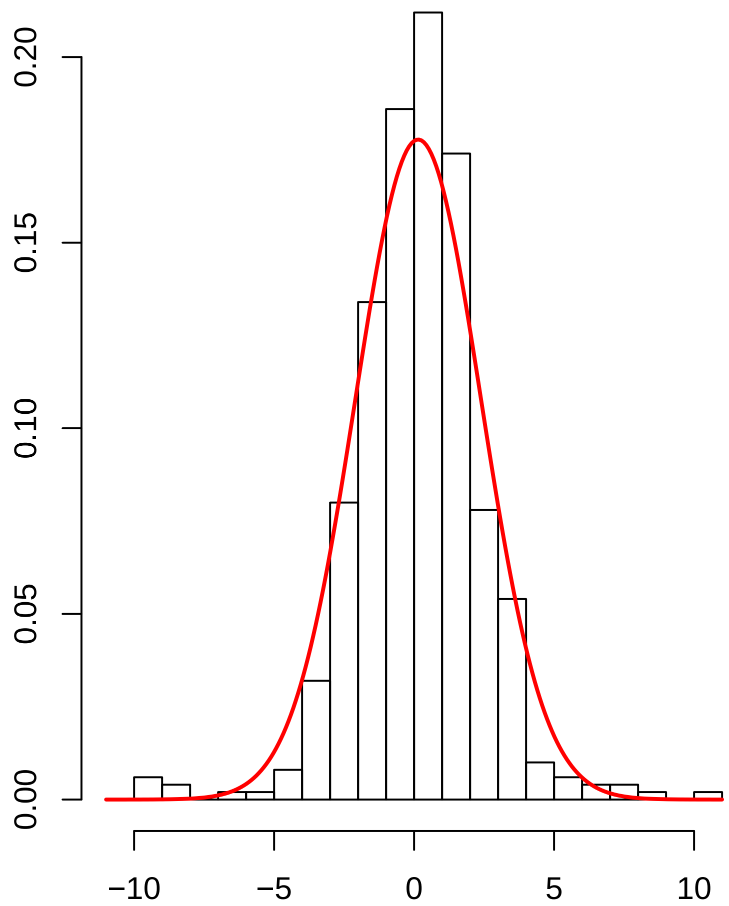
5.4 Simulation of a GARCH Process

Acf of the series and of the series of squares.



5.4 Simulation of a GARCH Process

Histogram of the distribution.



skewness:	−0.250
s.e.:	0.34
kurtosis:	3.428
s.e.:	0.68

5.5 Combining ARMA & GARCH

The purpose of combining ARMA and GARCH.

We have seen:

- ARMA models are conditional expectation models.
- Autocorrelation in the return series indicates that ARMA may be appropriate.
- GARCH models are conditional variance models.
- Autocorrelation in the *squared* return series indicates that GARCH may be appropriate.

5.5 Combining ARMA & GARCH

The purpose of combining ARMA and GARCH.

The conclusion is:

- If there is autocorrelation in the series as well as in the squared series, a combination of ARMA and GARCH may be appropriate.

5.5 Combining ARMA & GARCH

A mixed ARMA and GARCH model.

This is a process (X_t) defined as

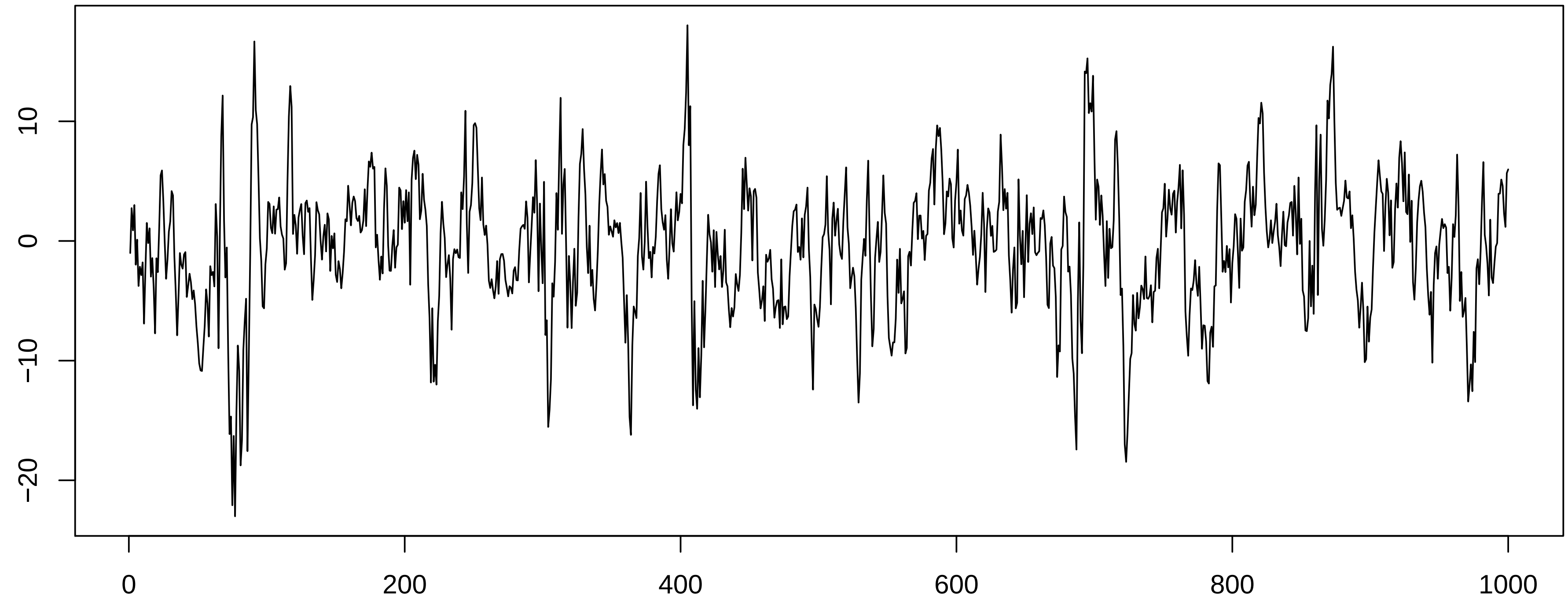
$$a(L)X_t = c + \beta(L)\epsilon_t,$$

where

- $a(L)$ is a polynomial of degree p in L,
- $\beta(L)$ is a polynomial of degree q in L,
- (ϵ_t) is a GARCH process.

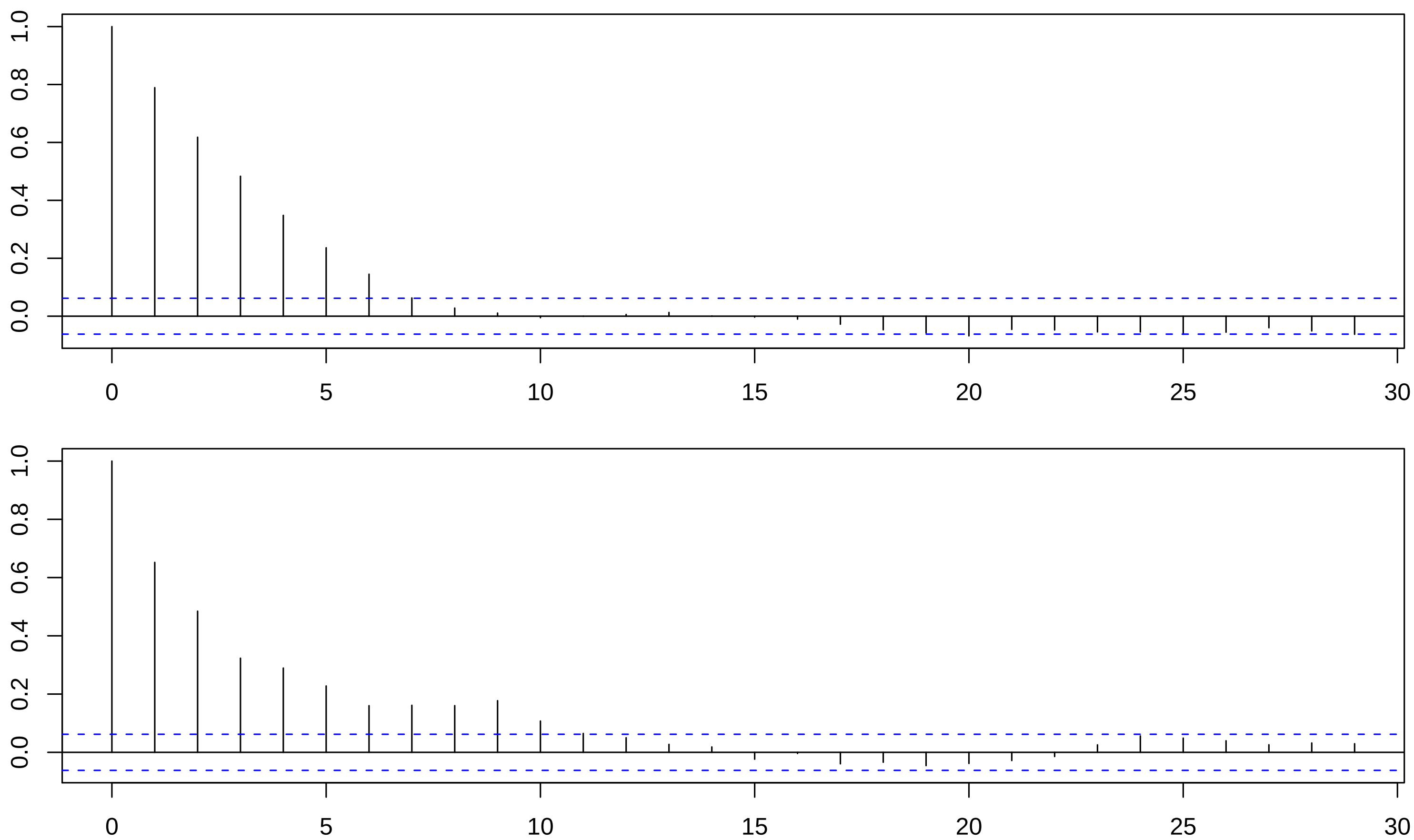
5.5 Combining ARMA & GARCH

Simulation of a mixed ARMA and GARCH model.



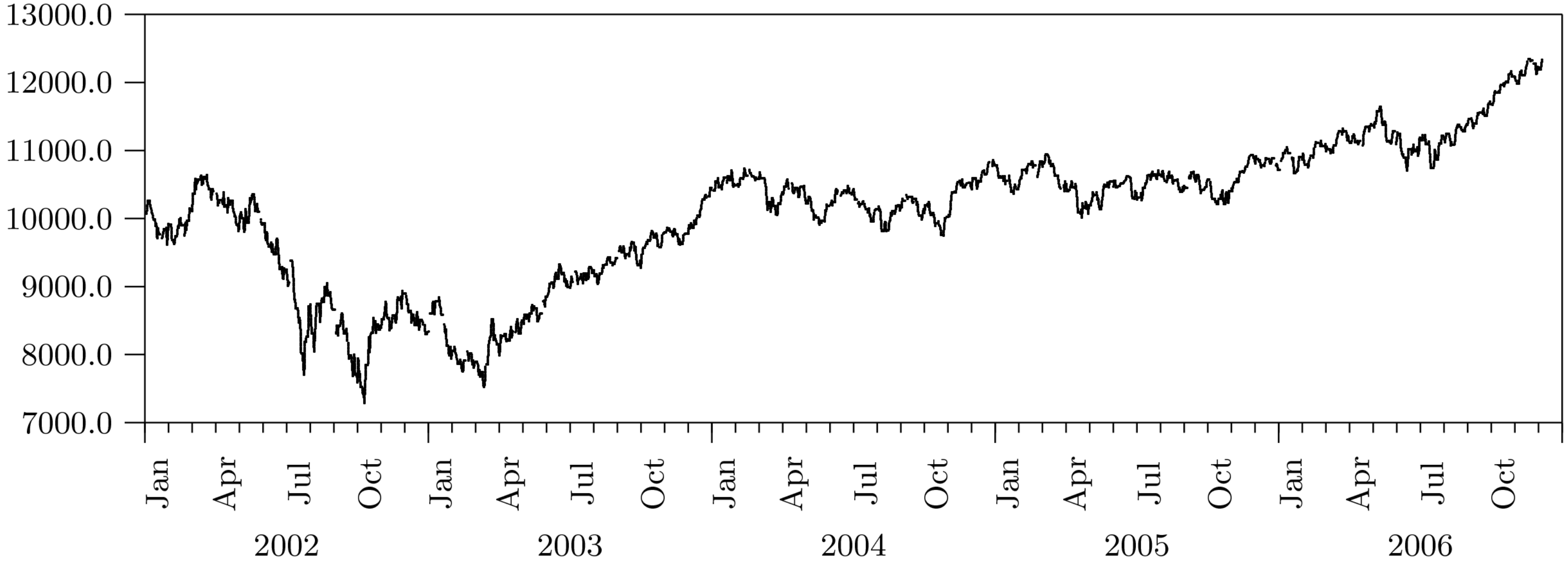
5.5 Combining ARMA & GARCH

Acfs of the simulated series and its squares.



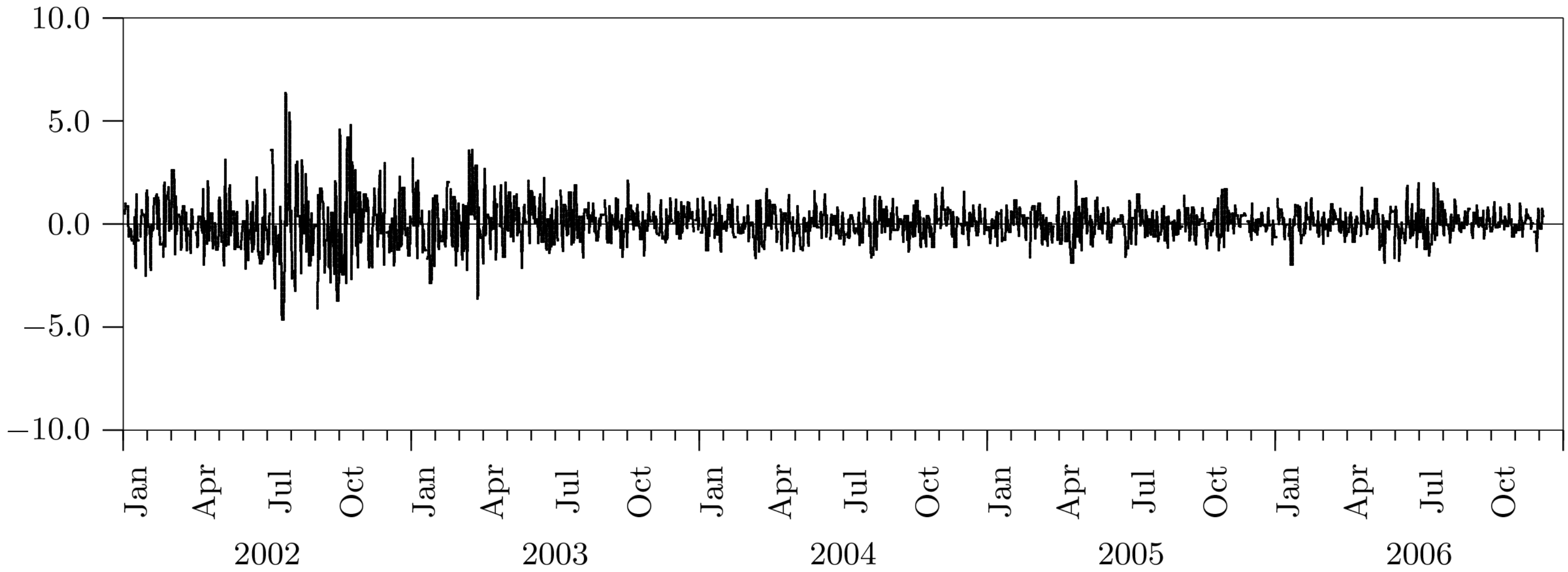
5.6 Examples

DJIA: the level series.



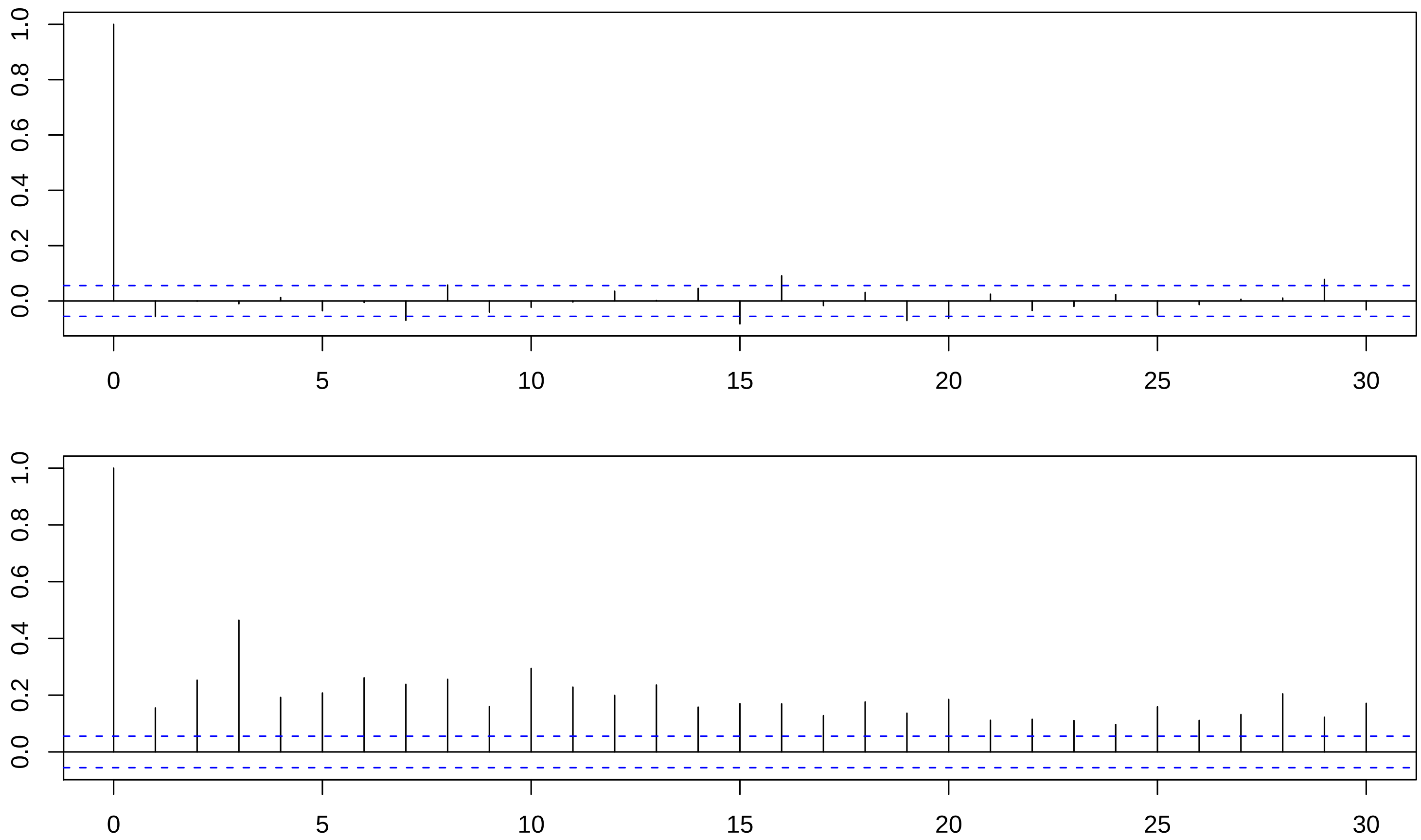
5.6 Examples

DJIA: the return series.



5.6 Examples

DJIA: acf of return series, squared return series.



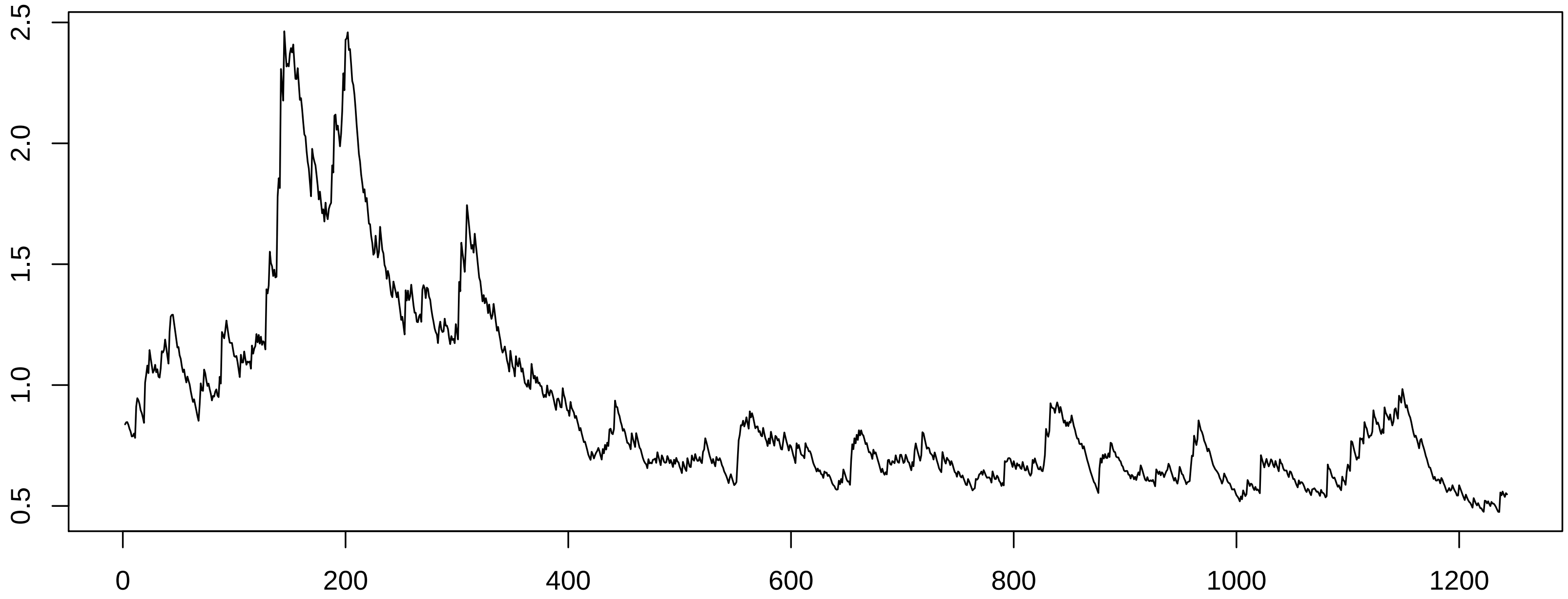
5.6 Examples

DJIA: fitting a GARCH model. (Mean return: 0.022)

```
Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0  0.005481    0.002682    2.044    0.041 *
a1  0.056106    0.009821    5.713 1.11e-08 ***
b1  0.936337    0.011614   80.619  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostic Tests:
        Jarque Bera Test

data:  Residuals
X-squared = 4.2215, df = 2, p-value = 0.1211

        Box-Ljung test

data:  Squared.Residuals
X-squared = 3.6251, df = 1, p-value = 0.05692
```

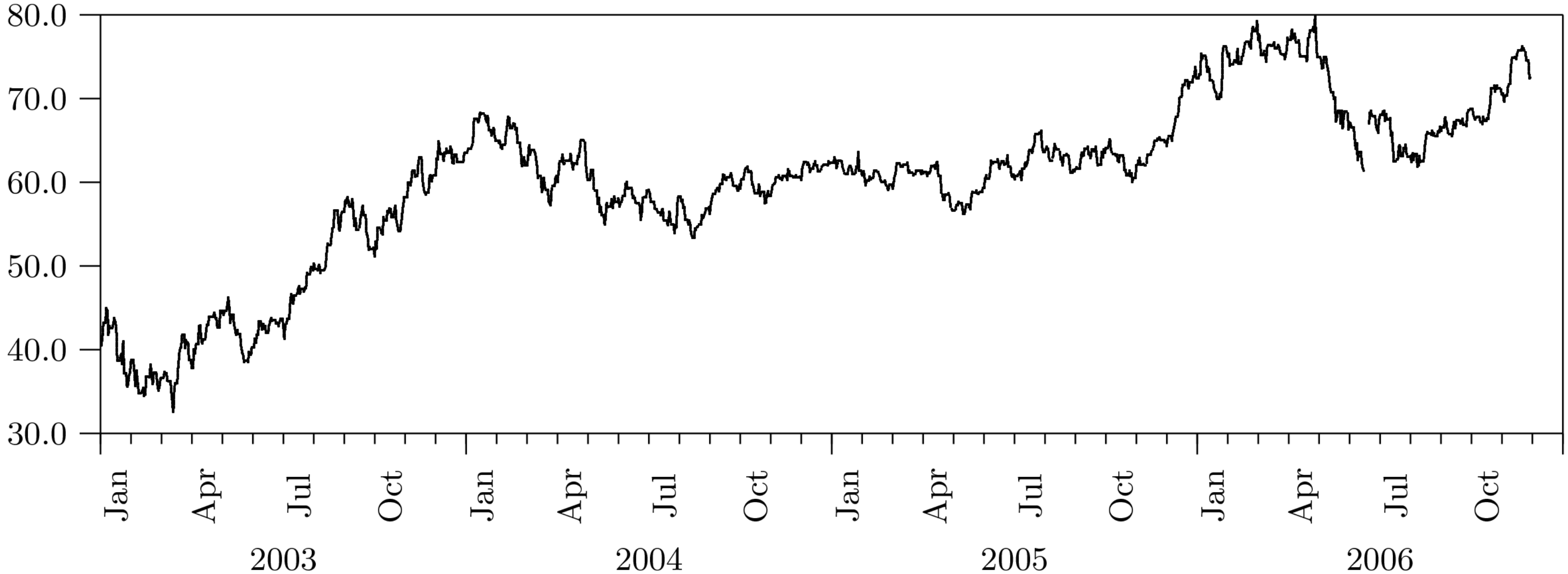
5.6 Examples

DJIA: conditional standard deviation series.



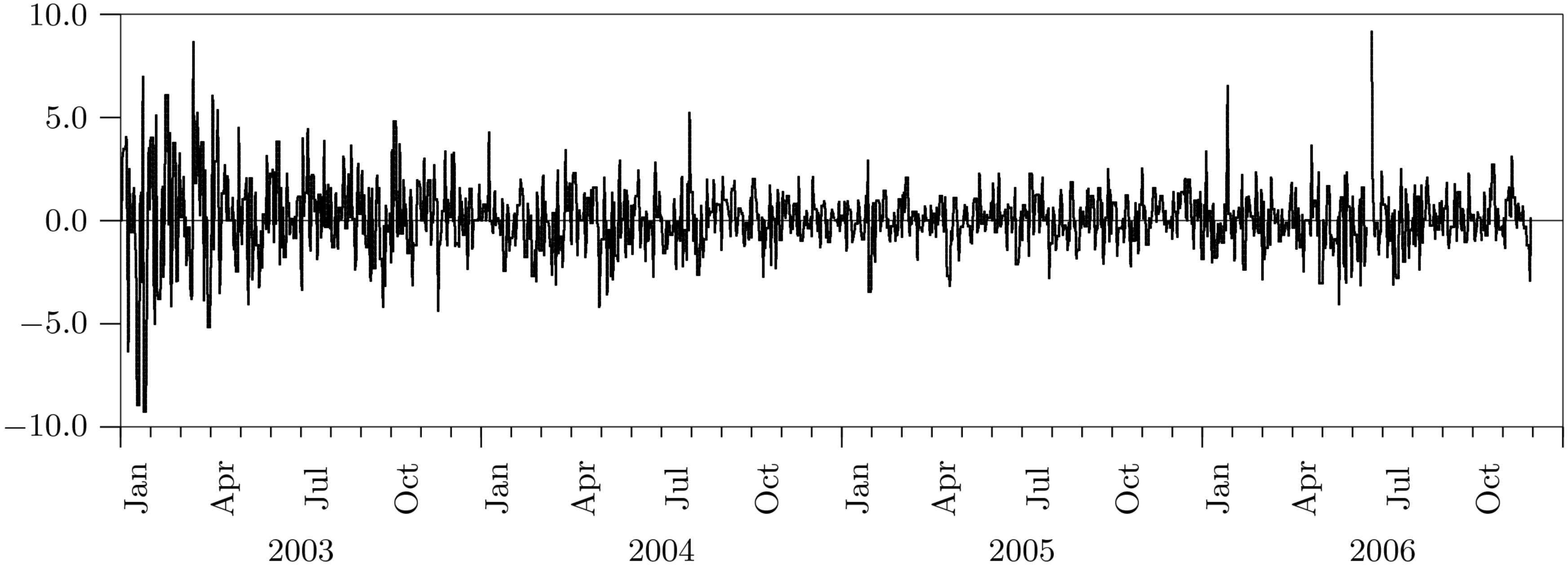
5.6 Examples

Siemens: the level series.



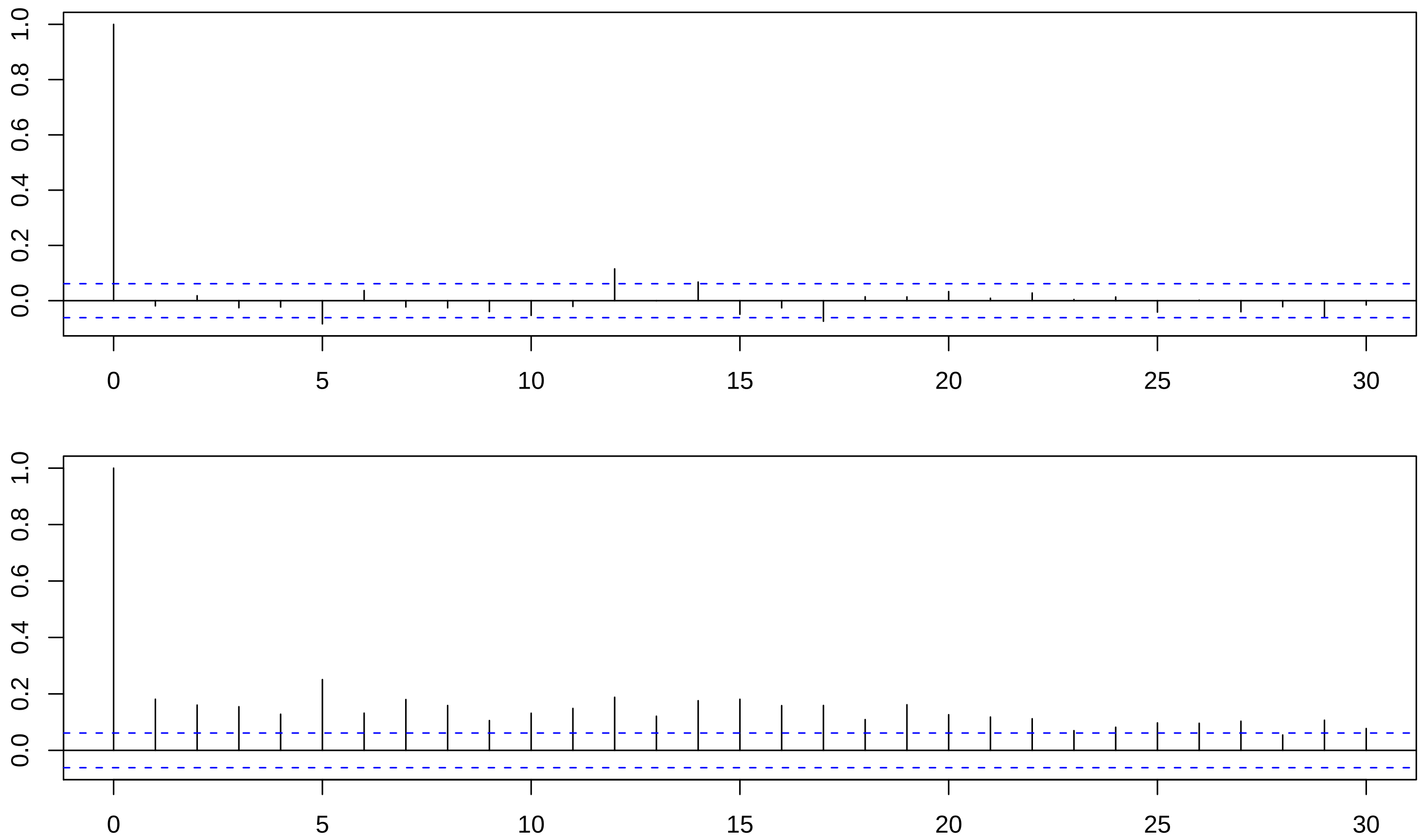
5.6 Examples

Siemens: the return series.



5.6 Examples

Siemens: acf of return series, squared return series.



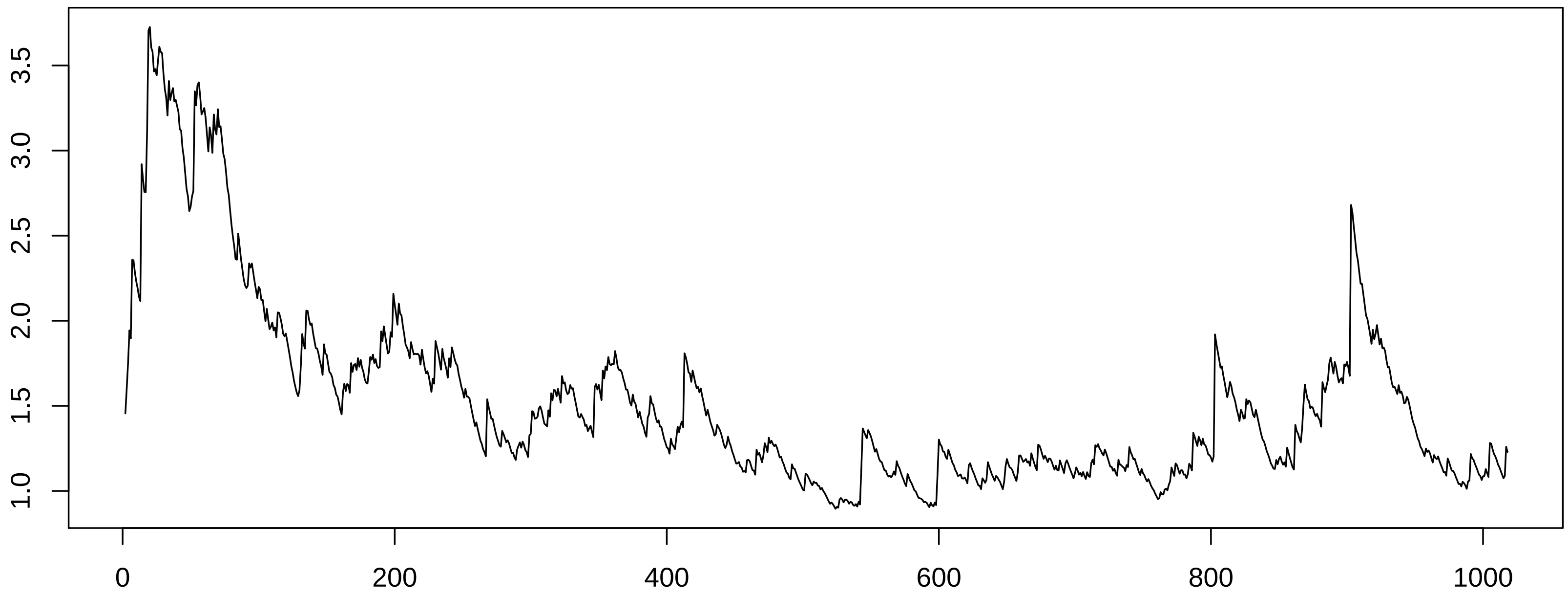
5.6 Examples

Siemens: fitting a GARCH model. (Mean return: 0.071)

```
Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0  0.030456    0.009857    3.090  0.00200 **
a1  0.056059    0.009051    6.193 5.89e-10 ***
b1  0.930380    0.011470   81.117  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostic Tests:
        Jarque Bera Test

data:  Residuals
X-squared = 214.8504, df = 2, p-value < 2.2e-16

        Box-Ljung test

data:  Squared.Residuals
X-squared = 0.0023, df = 1, p-value = 0.9617
```

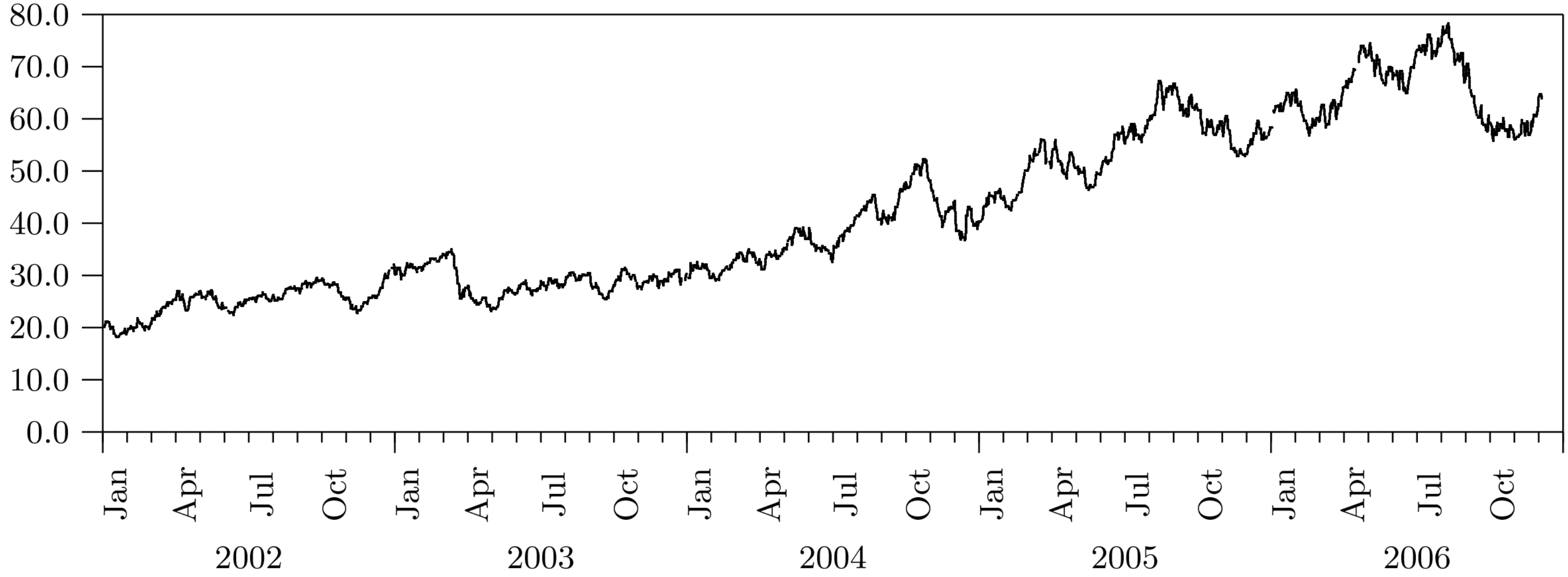
5.6 Examples

Siemens: conditional standard deviation series.



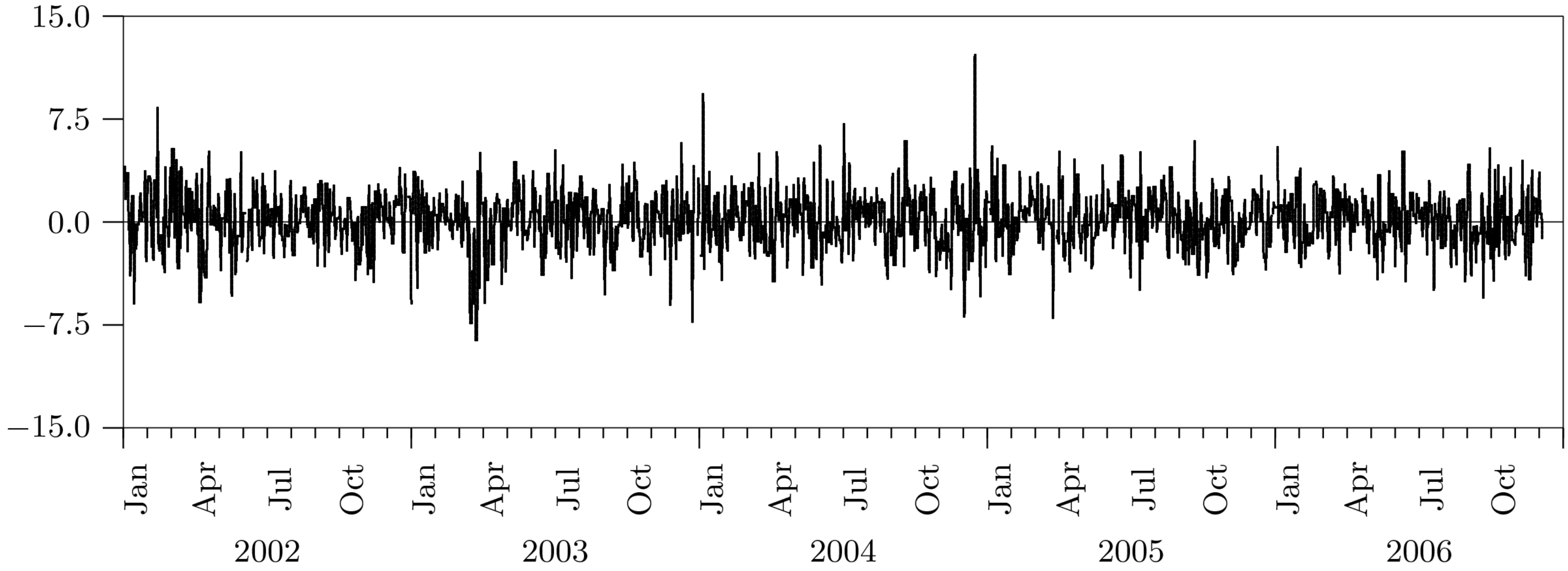
5.6 Examples

Brent crude oil: the level series.



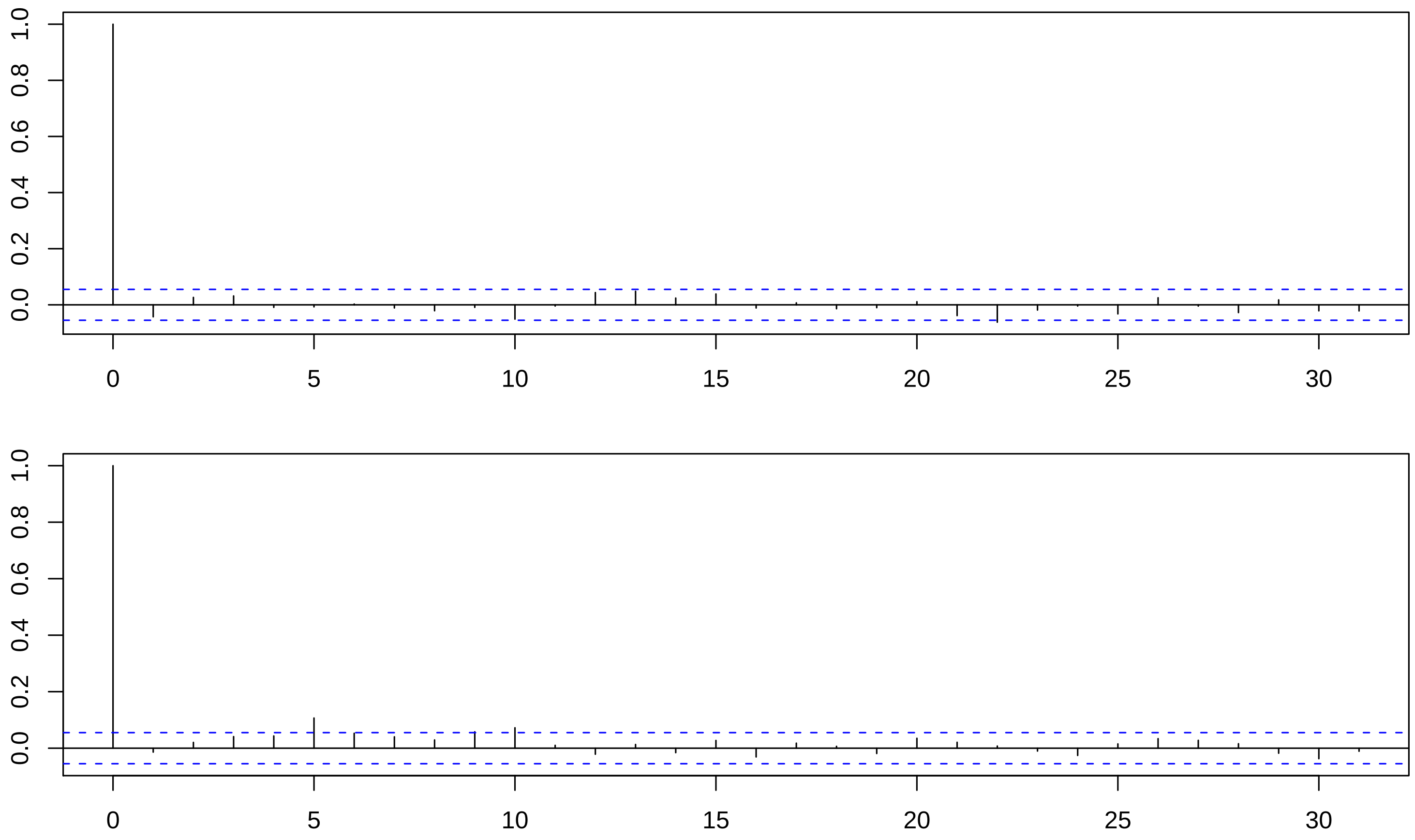
5.6 Examples

Brent crude oil: the return series.



5.6 Examples

Brent crude oil: acf of return series, squared return series.



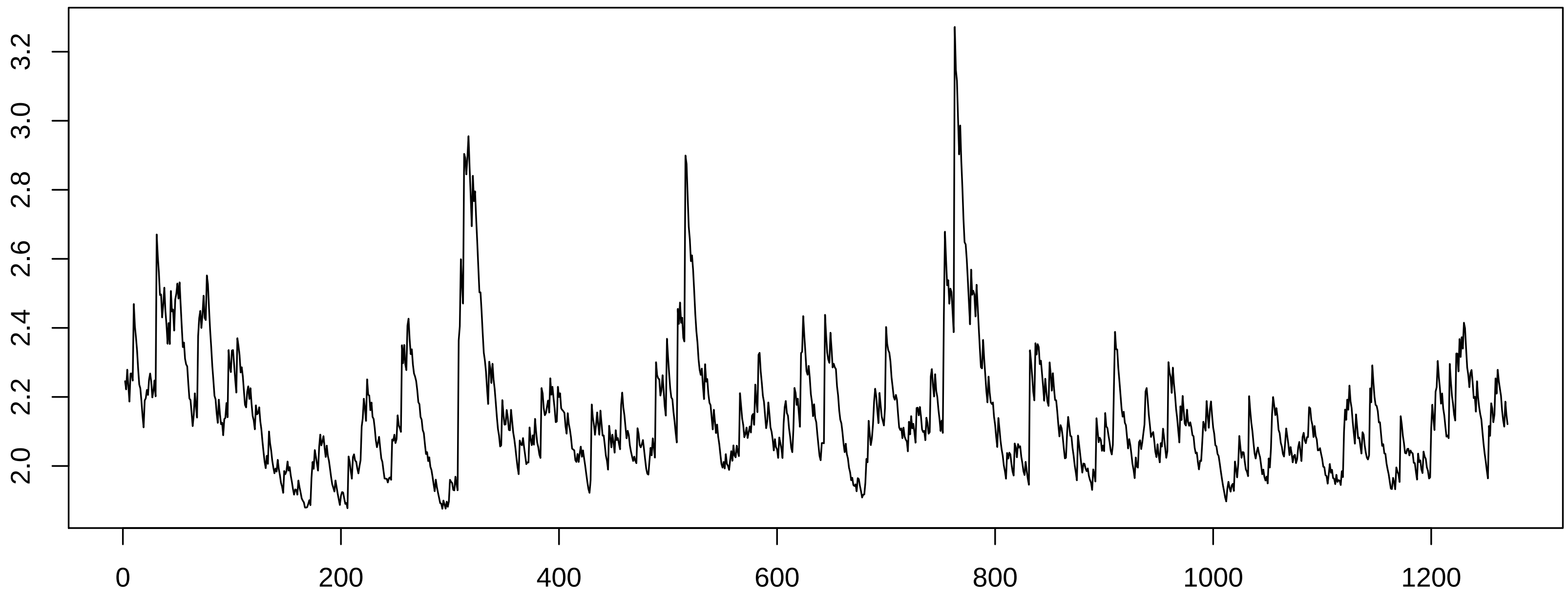
5.6 Examples

Brent crude oil: fitting a GARCH model. (Mean return: 0.117)

```
Coefficient(s):
    Estimate  Std. Error  t value Pr(>|t|)
a0   0.32744     0.15108    2.167  0.03021 *
a1   0.03889     0.01309    2.971  0.00297 **
b1   0.89014     0.04220   21.092  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Diagnostic Tests:
        Jarque Bera Test

data:  Residuals
X-squared = 62.6624, df = 2, p-value = 2.476e-14

        Box-Ljung test

data:  Squared.Residuals
X-squared = 2.3493, df = 1, p-value = 0.1253
```

5.6 Examples

Brent crude oil: conditional standard deviation series.



5.7 Limitations

This GARCH: symmetric & univariate.

Limitations of the GARCH processes we have seen are:

- "News impact" on the variance is symmetric:

$$\epsilon \mapsto \alpha_0 + \alpha_1 \epsilon^2 + \beta_1 \sigma^2$$

- This process is univariate.
  This limits its scope to investigate volatility spillovers.

5.7 Limitations

Example: Asymmetry in the DJIA return series.

- Is there empirical evidence for asymmetry in the series of returns?

- We can now do the following:
  - Make a scatterplot of r_t and r_{t+1}^2.
    (Our plot uses daily data , Apr 1981 through Sep 2008.)
  - Plot a local regression line. (R: `lowess`)
  - Look for asymmetry!
    Simulated GARCH data would produce a symmetric line.

5.7 Limitations

Example: Asymmetry in the DJIA return series.