

**EC 613:
Advanced Topics in Financial Econometrics**

**FEC 514:
Applications in Financial Modeling**

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- R files used for this course are available upon request.



Chapter 3:

VAR Models



3.1 VAR: Alternative Model Specifications

Structural (= “primitive”) form (lag 1, omitting the constants).

$$x_{1t} = -b_{12}x_{2t} + \gamma_{11}x_{1t-1} + \gamma_{12}x_{2t-1} + \epsilon_{1t}$$

$$x_{2t} = -b_{21}x_{1t} + \gamma_{21}x_{1t-1} + \gamma_{22}x_{2t-1} + \epsilon_{2t}$$

Equivalently:

$$B \cdot x_t = \Gamma_1 \cdot x_{t-1} + \epsilon_t$$

with

$$x_t = \begin{pmatrix} x_{1t} \\ x_{2t} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b_{12} \\ b_{21} & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{pmatrix},$$

(ϵ_t) : bivariate white noise, uncorrelated



3.1 VAR: Alternative Model Specifications

Questions etc.

- Questions:
 - interpretation of the model?
 - contemporaneous impact?
 - estimation of parameters?
- Purpose:
 - investigation of interaction
 - forecasting
 - assessment of spillovers



3.1 VAR: Alternative Model Specifications

Standard form.

Re-writing the model:

$$x_t = A_1 x_{t-1} + e_t$$

with

$$A_1 = B^{-1} \cdot \Gamma_1, \quad e_t = B^{-1} \cdot \epsilon_t$$

- this is called the “standard form”
- seemingly, no more contemporaneous impact
- errors are correlated
- estimation, identification?



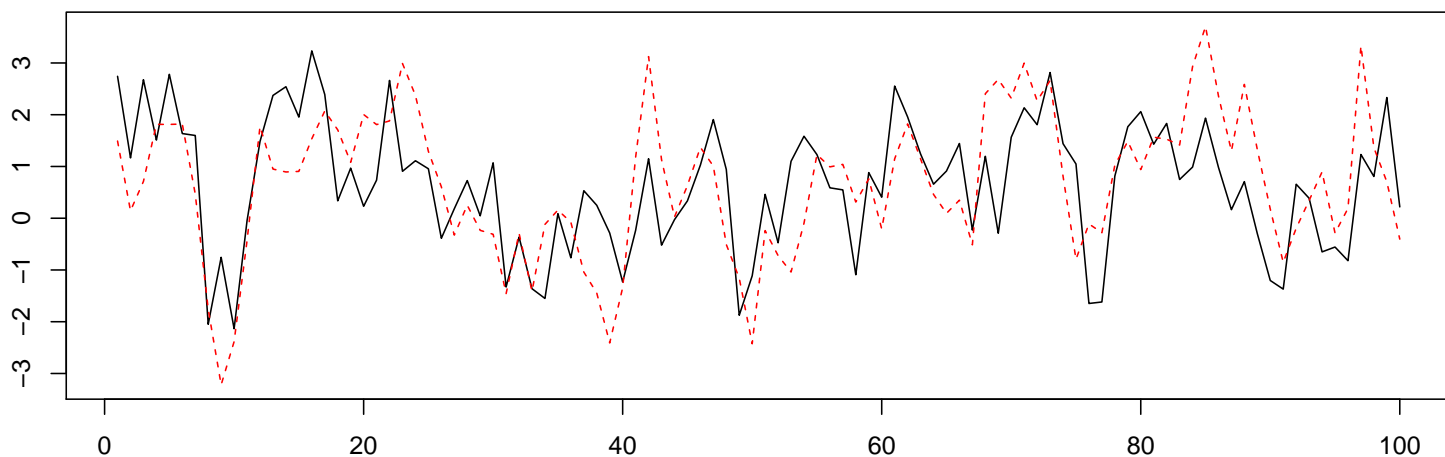
3.2 Estimation and Identification

A simulation example.

- Parameters:

$$B = \begin{pmatrix} 1 & -0.8 \\ 0.2 & 1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0.54 & -0.36 \\ 0.20 & 0.70 \end{pmatrix},$$

- Plot (part only):



3.2 Estimation and Identification

A simulation example.

Output when regressing x_{1t} on x_{2t} , x_{1t-1} , x_{2t-1} :

Call:

```
lm(formula = x1 ~ 0 + x2 + x1.lag1 + x2.lag1)
```

Residuals:

Min	1Q	Median	3Q	Max
-2.85820	-0.64004	0.00679	0.76603	2.80106

Coefficients:

	Estimate	Std. Error	t value	Pr(> t)
x2	0.55568	0.03691	15.06	< 2e-16 ***
x1.lag1	0.54674	0.02608	20.96	< 2e-16 ***
x2.lag1	-0.15130	0.04123	-3.67	0.000255 ***

Residual standard error: 1.006 on 997 degrees of freedom

Multiple R-squared: 0.5991, Adjusted R-squared: 0.5979

F-statistic: 496.6 on 3 and 997 DF, p-value: < 2.2e-16

correlation, x2 and eps1: -0.1072276



3.2 Estimation and Identification

The problems.

- VAR in structural form: cannot be estimated.
- Idea:
estimate standard form first, then find structural form from it.
- However: # parameters. . .
- Idea: restrict the primitive system. . .



3.2 Estimation and Identification

Structural form with $b_{21} = 0$ (Sims, 1980).

$$\begin{aligned}x_{1t} &= -b_{12}x_{2t} + \gamma_{11}x_{1t-1} + \gamma_{12}x_{2t-1} + \epsilon_{1t} \\x_{2t} &= \gamma_{21}x_{1t-1} + \gamma_{22}x_{2t-1} + \epsilon_{2t}\end{aligned}$$

Hence:

$$B = \begin{pmatrix} 1 & b_{12} \\ 0 & 1 \end{pmatrix}, \quad B^{-1} = \begin{pmatrix} 1 & -b_{12} \\ 0 & 1 \end{pmatrix}$$

The covariance matrix of e_t is:

$$\begin{pmatrix} \sigma_1^2 + b_{12}\sigma_2^2 & -b_{12}\sigma_2^2 \\ -b_{12}\sigma_2^2 & \sigma_2^2 \end{pmatrix}$$



3.2 Estimation and Identification

Remarks about the system when $b_{21} = 0$.

- The system is recursive.
- x_{1t} has no contemporaneous effect on x_{2t} .
- x_{2t} is called “causally prior” to x_{1t} .
- Both ϵ_{1t} and ϵ_{2t} affect the contemporaneous value of x_{1t} .
- Only ϵ_{2t} affects the contemporaneous value of x_{2t} .
- Observed e_{2t} are completely attributed to shocks to x_{2t} .



3.3 Impulse Response Functions

The case of AR(1).

- AR(1):

$$x_t = c + a x_{t-1} + \epsilon_t$$

- AR(1) as MA(∞):

$$x_t = \mu + \sum_{i=0}^{\infty} a^i \epsilon_{t-i} = \sum_{i=0}^{\infty} \Phi(i) \epsilon_{t-i}$$

- What is the effect of a one-unit shock in ϵ_t on $x_t, x_{t+1}, x_{t+2}, \dots$?



3.3 Impulse Response Functions

The case of a VAR.

- Standard form:

$$x_t = A_0 + A_1 x_{t-1} + e_t$$

- In MA(∞) representation:

$$x_t = \mu + \sum_{i=0}^{\infty} A_1^i e_{t-i} = \mu + \sum_{i=0}^{\infty} A_1^i B^{-1} \epsilon_{t-i} = \mu + \sum_{i=0}^{\infty} \Phi(i) \epsilon_{t-i}$$

- Here, $\Phi(i)$ is a matrix:

$$\Phi(i) = A_1^i \cdot B^{-1} = \begin{pmatrix} \Phi_{11}(i) & \Phi_{12}(i) \\ \Phi_{21}(i) & \Phi_{22}(i) \end{pmatrix}$$

- Effect of a one-unit shock in ϵ_t on $x_t, x_{t+1}, x_{t+2}, \dots$?



3.3 Impulse Response Functions

Impulse response functions.

- Each $i \mapsto \Phi_{jk}(i)$ is called an impulse response function.

- “impact multipliers”:

$$\Phi_{jk}(0)$$

- effect of a one-unit change in ϵ_{kt} on x_{jt+1} :

$$\Phi_{jk}(1)$$

- “long-run impact multipliers”, quantifying accumulated effects:

$$\sum_{i=0}^{\infty} \Phi_{jk}(i)$$



3.3 Impulse Response Functions

Example.

- Given:

$$A_1 = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix}, \quad \sigma_2^2 = \text{var}(e_{1t}) = \text{var}(e_{2t}), \quad \text{cor}(e_{1t}, e_{2t}) = 0.8$$

- Assumption: $b_{21} = 0$
- Then:

$$B = \begin{pmatrix} 1 & -0.8 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_1 = B \cdot A_1 = \begin{pmatrix} 0.54 & -0.36 \\ 0.20 & 0.70 \end{pmatrix}$$

- Decomposed errors:

$$e_{1t} = \epsilon_{1t} + 0.8\epsilon_{2t},$$

$$e_{2t} = \epsilon_{2t}$$



3.3 Impulse Response Functions

Example.

- The model is:

$$x_{1t} = 0.8x_{2t} + 0.54x_{1t-1} - 0.36x_{2t-1} + \epsilon_{1t}$$

$$x_{2t} = 0.20x_{1t-1} + 0.70x_{2t-1} + \epsilon_{2t}$$

or, equivalently:

$$x_{1t} = 0.70x_{1t-1} + 0.20x_{2t-1} + e_{1t}$$

$$x_{2t} = 0.20x_{1t-1} + 0.70x_{2t-1} + e_{2t}$$

- What is the effect of a shock in ϵ_{1t} or ϵ_{2t} on x_t, x_{t+1}, \dots ?



3.3 Impulse Response Functions

Example.

- time t (effect of a one-unit shock in ϵ_{2t} to x_{1t} and x_{2t}):
 - x_{2t} will jump by 1 unit
 - x_{1t} will jump by 0.8 units
- In terms of the impulse response function:

$$\Phi(0) = \mathbf{B}^{-1} = \begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix}$$



3.3 Impulse Response Functions

Example.

- time $t + 1$ (effect of a one-unit shock in ϵ_{2t} to x_{1t+1} and x_{2t+1}):

$$\begin{array}{l} x_{2t} = 0.20x_{1t-1} + 0.70x_{2t-1} + e_{2t} \\ \text{will jump by: } \quad 0.20 \cdot 0.80 + 0.70 \cdot 1 \quad \quad \quad = 0.86 \end{array}$$

$$\begin{array}{l} x_{1t} = 0.70x_{1t-1} + 0.20x_{2t-1} + e_{1t} \\ \text{will jump by: } \quad 0.70 \cdot 0.80 + 0.20 \cdot 1 \quad \quad \quad = 0.76 \end{array}$$

- In terms of the impulse response function:

$$\Phi(1) = A_1 \cdot B^{-1} = \begin{pmatrix} 0.7 & 0.2 \\ 0.2 & 0.7 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0.8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.70 & 0.76 \\ 0.20 & 0.86 \end{pmatrix}$$

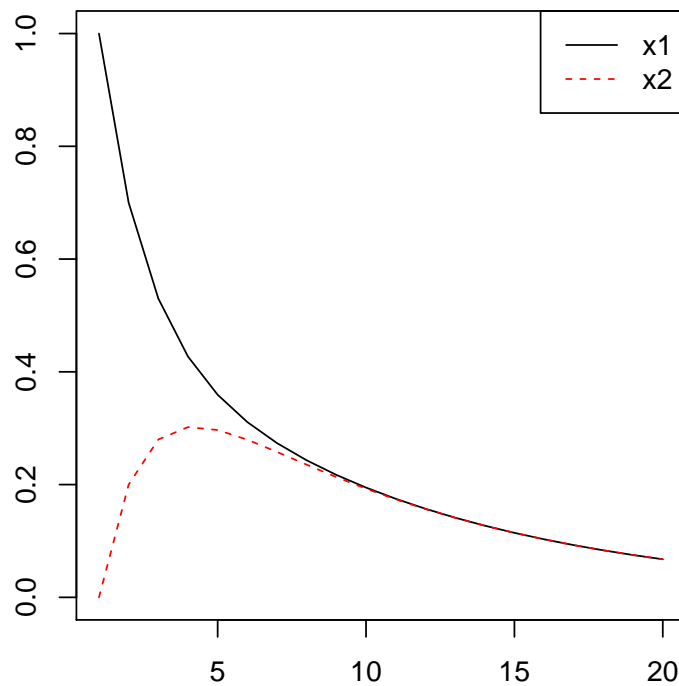


3.3 Impulse Response Functions

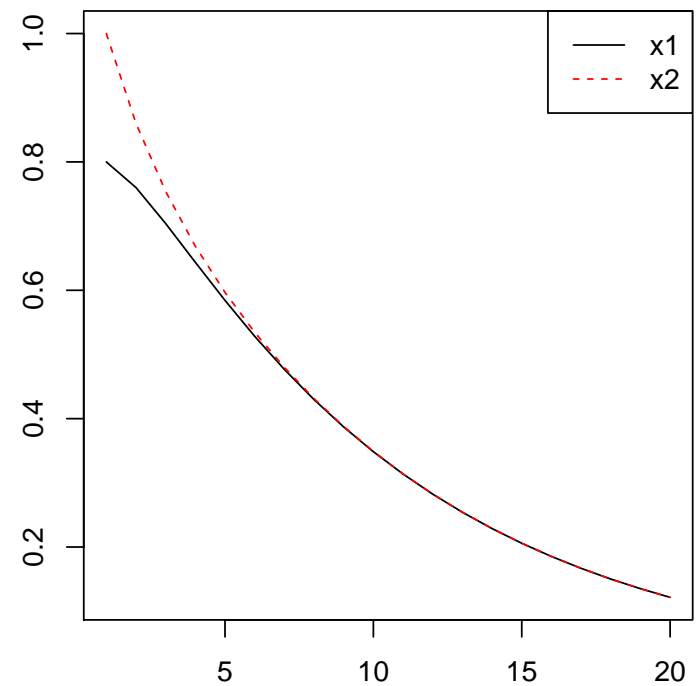
Example.

- impulse response functions:

of shock in ϵ_{1t}



of shock in ϵ_{2t}



3.3 Impulse Response Functions

Decompositions / orderings.

- Example: We assumed $b_{21} = 0$.
What if we had assumed $b_{12} = 0$?
- A decomposition

$$\begin{aligned}e_{1t} &= \epsilon_{1t} - b_{12}\epsilon_{2t}, \\e_{2t} &= \epsilon_{2t}\end{aligned}$$

is called a ordering of the variables. (x_{2t} , causally prior)



3.3 Impulse Response Functions

Aspects when deciding which ordering might be appropriate.

- A priori knowledge of contemporaneous effects.
- Importance of ordering depends on the magnitude of $|\text{cor}(e_{1t}, e_{2t})|$.
 - $|\text{cor}(e_{1t}, e_{2t})| = 0$: ordering is immaterial.
 - $|\text{cor}(e_{1t}, e_{2t})| = 1$: single shock affects both variables contemporaneously.
 - Rule of thumb: If $|\text{cor}(e_{1t}, e_{2t})| > 0.2$ with 100 observations, there is significant correlation.
Try different orderings, check for plausibility.
- . . . or use other means to identify the structural model!



3.4 Forecast Error Variance Decomposition

Forecasts and forecast errors.

- Given A_0 , A_1 , x_t , forecast x_{t+i} !

$i = 1$:

forecast $\hat{x}_{t+1} = A_0 + A_1 x_t$

forecast error e_{t+1}

$i = 2$:

forecast $\hat{x}_{t+2} = (1 + A_1) A_0 + A_1^2 x_t$

forecast error $e_{t+2} + A_1 e_{t+1}$

$i = n$:

forecast $\hat{x}_{t+n} = (1 + A_1 + \dots + A_1^{n+1}) A_0 + A_1^n x_t$

forecast error $e_{t+n} + A_1 e_{t+n-1} + A_1^2 e_{t+n-2} + \dots + A_1^{n-1} e_{t+1}$



3.4 Forecast Error Variance Decomposition

The forecast error in terms of the structural model.

$$\sum_{i=0}^{n-1} A_1^i e_{t+n-i} = \sum_{i=0}^{n-1} A_1^i B^{-1} \epsilon_{t+n-i} = \sum_{i=0}^{n-1} \Phi(i) \epsilon_{t+n-i}$$

For x_{1t+n} , the forecast error will be:

$$\begin{aligned} & \sum_{i=0}^{n-1} (\Phi_{11}(i) \quad \Phi_{12}(i)) \begin{pmatrix} \epsilon_{1t+n-i} \\ \epsilon_{2t+n-i} \end{pmatrix} \\ &= \begin{matrix} \Phi_{11}(0) & \epsilon_{1t+n} & + & \Phi_{12}(0) & \epsilon_{2t+n} & + \\ & \vdots & & \vdots & & \\ \Phi_{11}(n-1) & \epsilon_{1t+1} & + & \Phi_{12}(n-1) & \epsilon_{2t+1} \end{matrix} \end{aligned}$$



3.4 Forecast Error Variance Decomposition

Decomposition of the forecast error variance.

For x_{1t+n} , the forecast error variance is:

$$\begin{aligned}
 & \text{var} \left(\sum_{i=0}^{n-1} (\Phi_{11}(i) \quad \Phi_{12}(i)) \begin{pmatrix} \epsilon_{1t+n-i} \\ \epsilon_{2t+n-i} \end{pmatrix} \right) \\
 &= \begin{matrix} \Phi_{11}^2(0) & \sigma_1^2 & + & \Phi_{12}^2(0) & \sigma_2^2 & + \\ & \vdots & & \vdots & & \\ \Phi_{11}^2(n-1) & \sigma_1^2 & + & \Phi_{12}^2(n-1) & \sigma_2^2 & \end{matrix} \\
 &= \underbrace{\sigma_1^2 \cdot \sum_{i=0}^{n-1} \Phi_{11}^2(i)}_{\text{due to . . . shocks in } \epsilon_{1t}} + \underbrace{\sigma_2^2 \cdot \sum_{i=0}^{n-1} \Phi_{12}^2(i)}_{\text{shocks in } \epsilon_{2t}}
 \end{aligned}$$



3.4 Forecast Error Variance Decomposition

Spillover table and spillover index.

- Schematically:

		from. . .	
		ϵ_{1t}	ϵ_{2t}
to. . .	x_{1t+n}	□	■
	x_{2t+n}	■	□

- Each row of the spillover table defines a fevd.

- spillover index = $\frac{\sum \blacksquare}{\sum \blacksquare + \sum \square}$ (Diebold & Yilmaz, 2009)



3.4 Forecast Error Variance Decomposition

Example.

For x_{1t+n} , the forecast error variance decomposition (fevd) is:

n	from. . .		from. . .	
	ϵ_{1t} (absolute)	ϵ_{2t}	ϵ_{1t} (relative)	ϵ_{2t}
1	1.000	0.800	0.556	0.444
2	1.281	1.296	0.497	0.503
3	1.463	1.710	0.461	0.539
4	1.592	2.051	0.437	0.563
5	1.689	2.330	0.420	0.580
\vdots	\vdots	\vdots	\vdots	\vdots
∞	2.068	3.531	0.369	0.631

