

**EC 613:
Advanced Topics in Financial Econometrics**

**FEC 514:
Applications in Financial Modeling**

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 İSTANBUL BİLGİ ÜNİVERSİTESİ

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Chapter 3:

Univariate ARMA Models



3.1 Introduction

ARMA models: scope and outlook.

- ARMA models are a class of stochastic processes.
- The idea behind ARMA models is:

Exploit the autocorrelation structure of the series!

- In this chapter, we shall see some properties and the limitations of ARMA processes in financial modeling.



3.1 Introduction

The simplest ARMA models.

- MA(1): $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$
- AR(1): $X_t = c + aX_{t-1} + \epsilon_t$
- ARMA(1,1): $X_t = c + aX_{t-1} + \epsilon_t + \beta\epsilon_{t-1}$

(ϵ_t) is white noise.



3.2 MA Processes

MA: Definition.

An MA(q) process (X_t) is defined as

$$X_t = c + \epsilon_t + \beta_1\epsilon_{t-1} + \dots + \beta_q\epsilon_{t-q}$$

or, equivalently,

$$X_t = c + \beta(L)\epsilon_t,$$

where $\beta(L)$ is a polynomial of degree q in L .

- L is the lag operator.
- (ϵ_t) is white noise.



3.2 MA Processes

MA(1): Its autocorrelation function.

For an MA(1) process: $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$

$$\begin{aligned}\text{var}(X_t) &= \text{var}(\epsilon_t + \beta\epsilon_{t-1}) = (1 + \beta^2)\sigma_\epsilon^2, \\ \text{cov}(X_t, X_{t+1}) &= \text{cov}(\epsilon_t + \beta\epsilon_{t-1}, \epsilon_{t+1} + \beta\epsilon_t) = \beta\sigma_\epsilon^2, \\ \text{cov}(X_t, X_{t+s}) &= 0 \text{ for } s \geq 2.\end{aligned}$$

The acf is therefore:

$$s \mapsto \rho(s) = \begin{cases} \beta/(1 + \beta^2) & \text{for } s = 1, \\ 0 & \text{for } s \geq 2 \end{cases}$$

The acf of any MA process cuts off.



3.2 MA Processes

MA(1): Its unconditional moments.

For an MA(1) process $X_t = c + \epsilon_t + \beta\epsilon_{t-1}$:

$$E(X_t) = c,$$

$$\text{var}(X_t) = (1 + \beta^2)\sigma_\epsilon^2.$$



3.2 MA Processes

“Inverting” an MA(1) process.

For an MA(1) process $X_t = \epsilon_t + \beta\epsilon_{t-1}$, we can write:

$$X_t = (1 + \beta L)\epsilon_t,$$

$$\epsilon_t = \frac{X_t}{1 + \beta L},$$

$$\epsilon_t = \sum_{s=0}^{\infty} (-\beta)^s X_{t-s},$$

$$X_t = \epsilon_t + \beta \sum_{s=0}^{\infty} (-\beta)^s X_{t-s-1}.$$



3.2 MA Processes

MA(1): Its conditional moments.

Therefore:

$$\begin{aligned} \mathbb{E}(X_t | X_{t-1}, \dots) &= \beta \sum_{s=0}^{\infty} (-\beta)^s X_{t-s-1}, \\ \text{var}(X_t | X_{t-1}, \dots) &= \sigma_{\epsilon}^2. \end{aligned}$$

We observe:

- An MA process can be forecast only if it is invertible.
- MA is a conditional expectation model.



3.3 AR Processes

AR: Definition. An AR(p) process (X_t) is defined as

$$X_t = c + a_1 X_{t-1} + \dots + a_p X_{t-p} + \epsilon_t$$

or, equivalently,

$$a(L)X_t = c + \epsilon_t,$$

where $a(L)$ is a polynomial of degree p in L , such that (X_t) is stationary.

- L is the lag operator.
- (ϵ_t) is white noise.



3.3 AR Processes

AR(1): Its autocorrelation function.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$, we can write:

$$\begin{aligned}(1 - aL)X_t &= c + \epsilon_t, \\ X_t &= \frac{c}{1 - aL} + \frac{\epsilon_t}{1 - aL}, \\ X_t &= \frac{c}{1 - a} + \sum_{s=0}^{\infty} a^s \epsilon_{t-s}.\end{aligned}$$

This can be used to show that the acf of this process is

$$s \mapsto \rho(s) = a^s.$$



3.3 AR Processes

AR(1): Its unconditional moments.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$:

$$E(X_t) = \frac{c}{1-a},$$

$$\text{var}(X_t) = \frac{\sigma_\epsilon^2}{1-a^2}.$$



3.3 AR Processes

AR(1): Its conditional moments.

For an AR(1) process $X_t = c + aX_{t-1} + \epsilon_t$:

$$\mathbb{E}(X_t | X_{t-1}, \dots) = c + aX_{t-1},$$

$$\text{var}(X_t | X_{t-1}, \dots) = \sigma_\epsilon^2.$$

As for an MA process, we observe:

- AR is a conditional expectation model.



3.4 Mixed AR/MA Processes

ARMA processes.

A mixed model

$$a(L)X_t = c + \beta(L)\epsilon_t$$

where $a(L)$ is a polynomial of degree p in L and $\beta(L)$ is a polynomial of degree q in L is called an ARMA(p, q) process, provided that it is stationary.



3.4 Mixed AR/MA Processes

ARIMA processes.

Consider a process (X_t) such that

$$Z_t = (1 - L)X_t = X_t - X_{t-1}$$

is an ARMA(p, q) process.

Then, (X_t) is called an ARIMA($p, 1, q$) process.

Keywords:

- stochastic trend,
- differencing.



3.5 Seasonal ARMA Models

Consider the case of monthly data.

A simple multiplicative seasonal model is:

$$(1 - a_{12}L^{12})(1 - a_1L)X_t = c + (1 - \beta_{12}L^{12})(1 - \beta_1L)\epsilon_t$$

This is called a SARMA(1,1) \times (1,1)₁₂ process.

Observe that there is, in particular, a direct impact of X_{t-13} on X_t .



3.6 Tentative Identification

Goals, aspects, tools of model identification.

- Identification means: Find a stochastic model (here: an AR(I)MA model) which may have created the observed series.
- Most important tools in model identification: acf / pacf
- The procedure is:
 - Determine the empirical acf / pacf.
 - Find an ARMA process with similar acf / pacf.



3.6 Tentative Identification

Goals, aspects, tools of model identification.

Considerations:

- The model should be simple (“parsimonious”).
- The residuals should have no more autocorrelation structure. (They should be white noise. — What does this mean?)



3.6 Tentative Identification

Concerning model simplicity:

The Akaike information criterion (AIC). It is computed as:

$$\text{AIC} = T \cdot \ln(\text{residual sum of squares}) + 2n,$$

where:

- n = number of parameters estimated (typically, $p + q + 1$),
- T = number of usable observations.

- A model should be selected such that AIC becomes small.
- AIC penalizes the use of additional parameters.



3.6 Tentative Identification

Concerning the residuals:

The Box-Ljung statistic.

It permits to test the null hypothesis

H_0 : There is no autocorrelation in the residuals up to lag s .

against the alternative

H_1 : There IS autocorrelation in the residuals up to lag s .



3.6 Tentative Identification

Concerning the residuals:

The Box-Ljung statistic.

It is defined as

$$Q = T(T + 2) \sum_{k=1}^s \frac{r_k^2}{T - k},$$

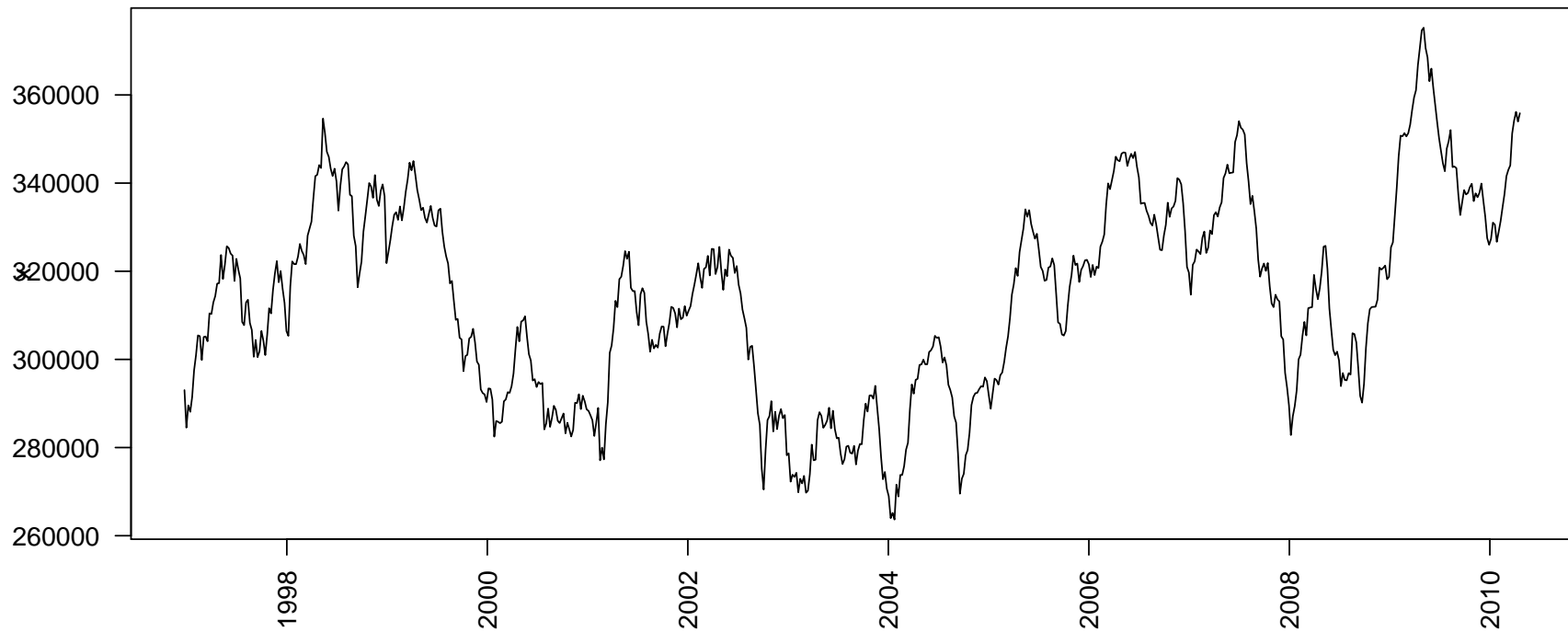
where r_k is the empirical autocorrelation of the residuals.

Critical: large values.



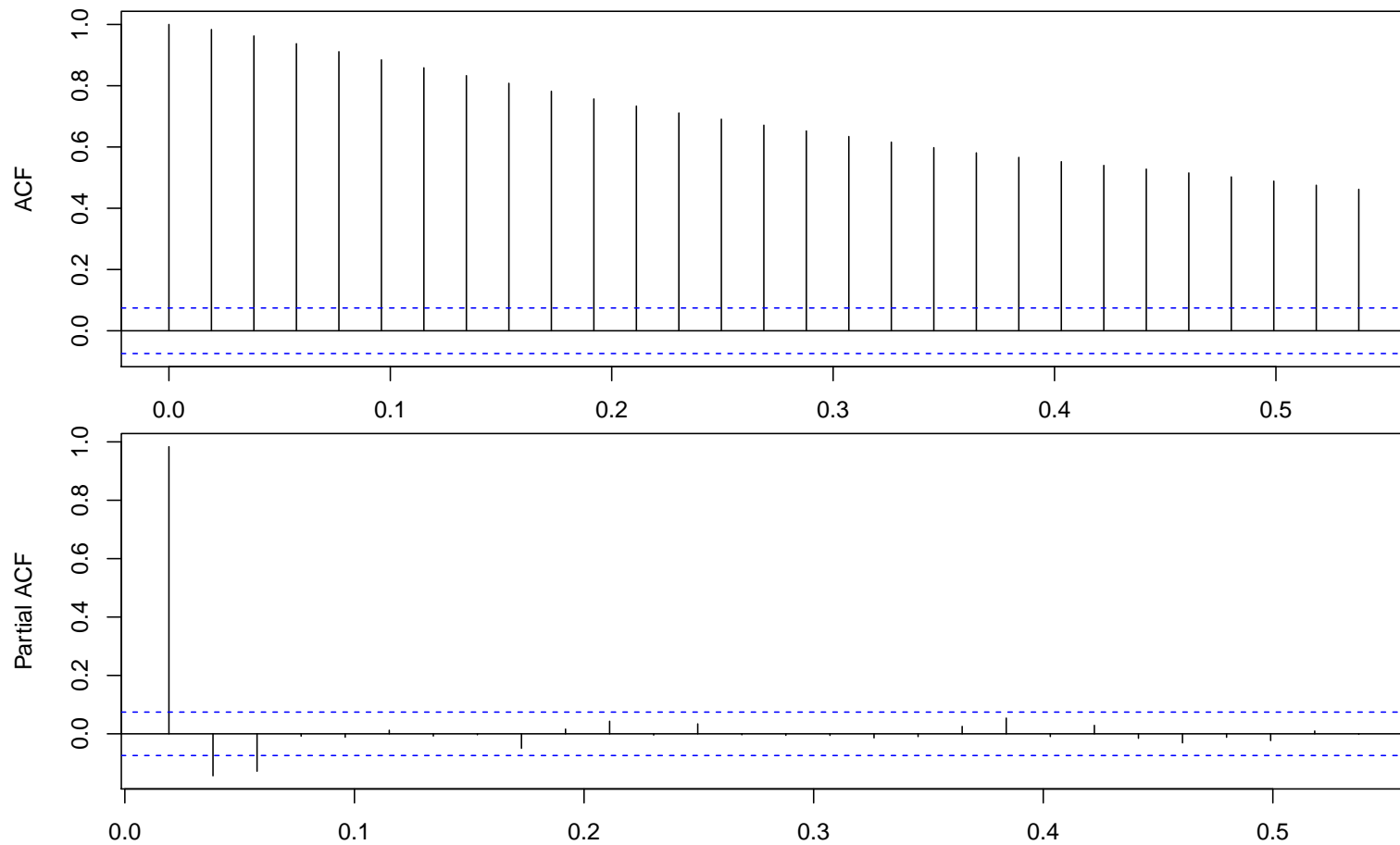
3.7 Example: Crude Oil Inventories

US crude oil inventories in 1000 barrels, reported weekly.



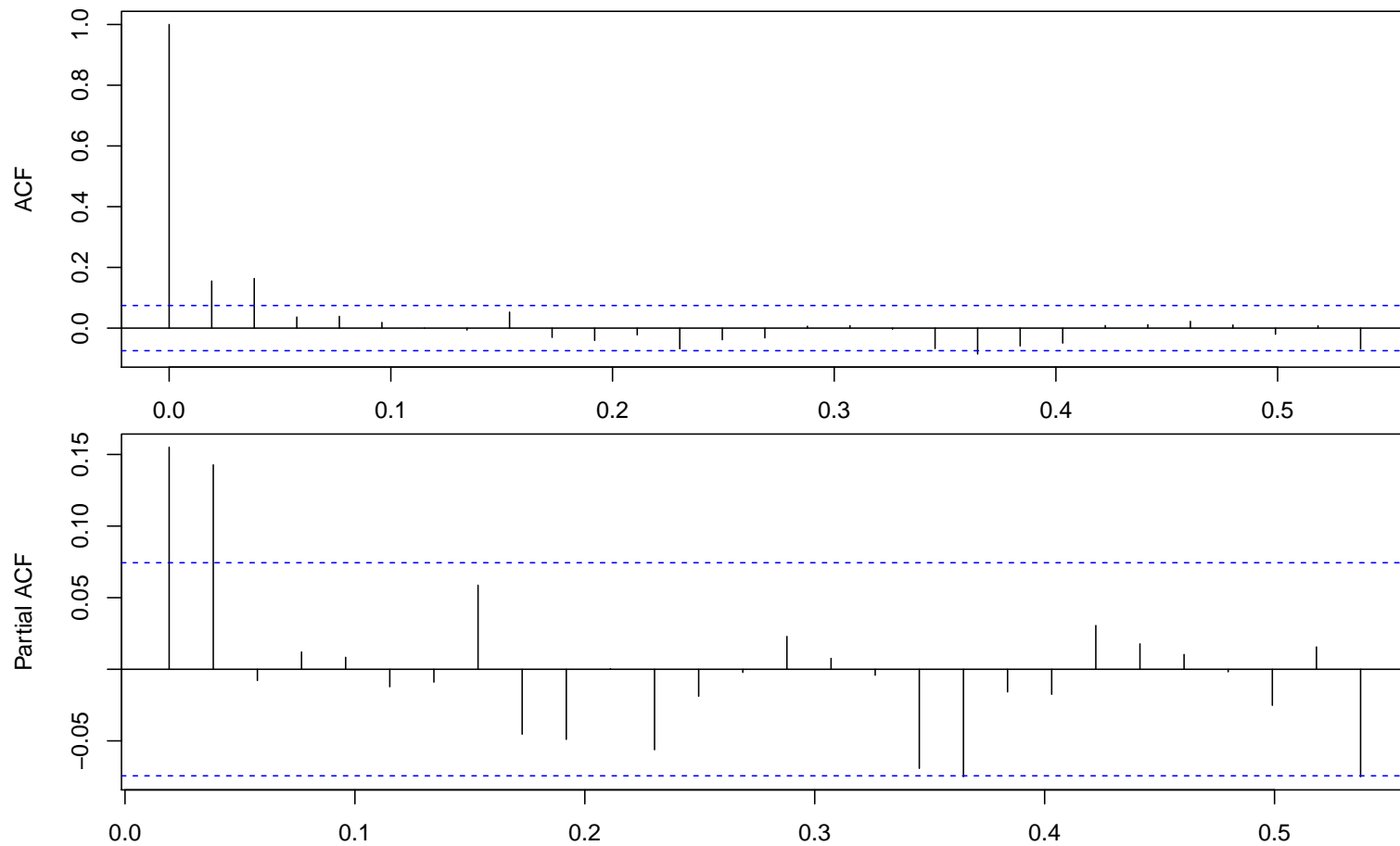
3.7 Example: Crude Oil Inventories

Acf, pacf of the series.



3.7 Example: Crude Oil Inventories

Acf, pacf of the *differenced* series.



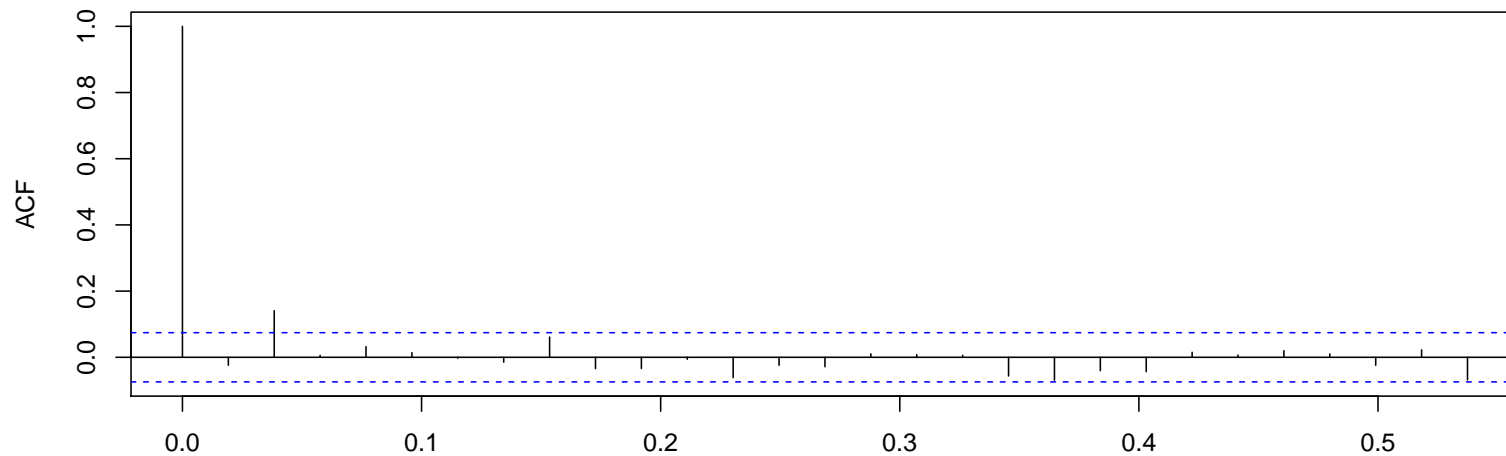
3.7 Example: Crude Oil Inventories

1st trial: ARIMA(1,1,0). Estimation and acf of residual series.

Coefficients:

```
      ar1  
      0.1567  
s.e. 0.0376
```

```
sigma^2 estimated as 13238991:  log likelihood = -6675.1,  aic = 13354.19
```



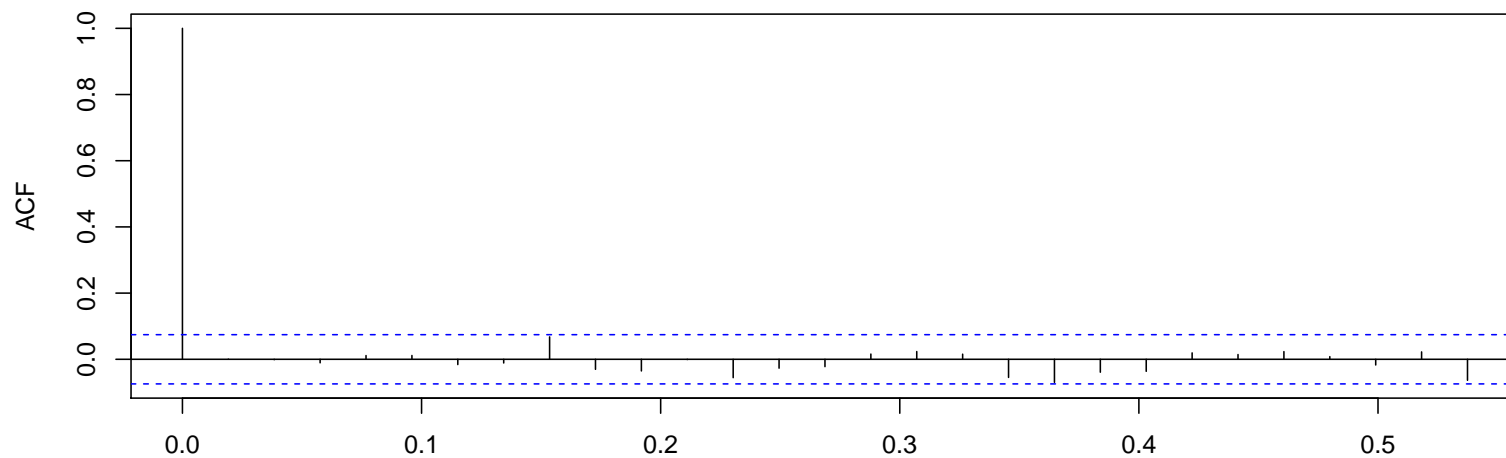
3.7 Example: Crude Oil Inventories

2nd trial: ARIMA(2,1,0). Estimation and acf of residual series.

Coefficients:

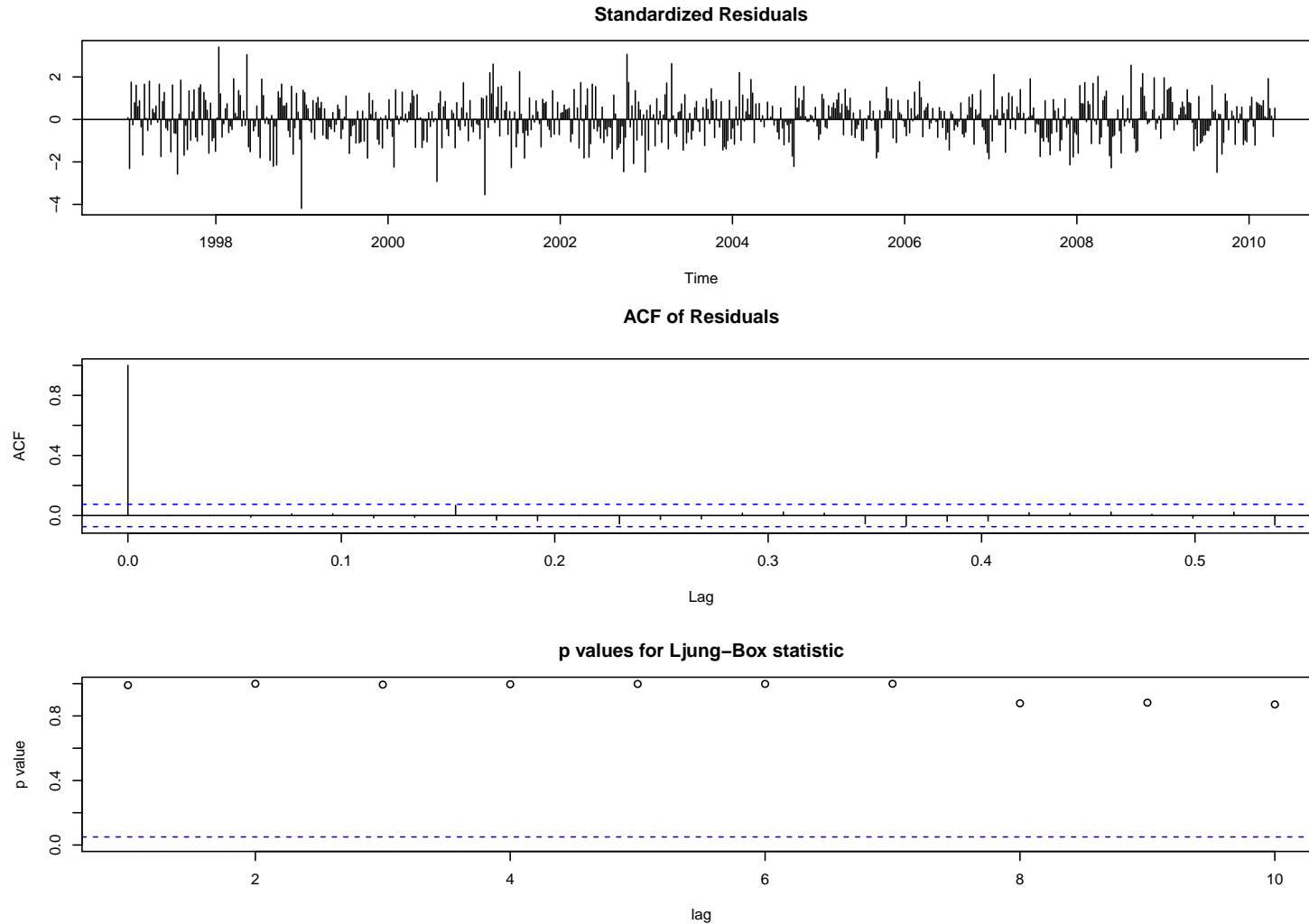
	ar1	ar2
	0.1334	0.1436
s.e.	0.0377	0.0378

σ^2 estimated as 12967888: log likelihood = -6667.94, aic = 13341.88



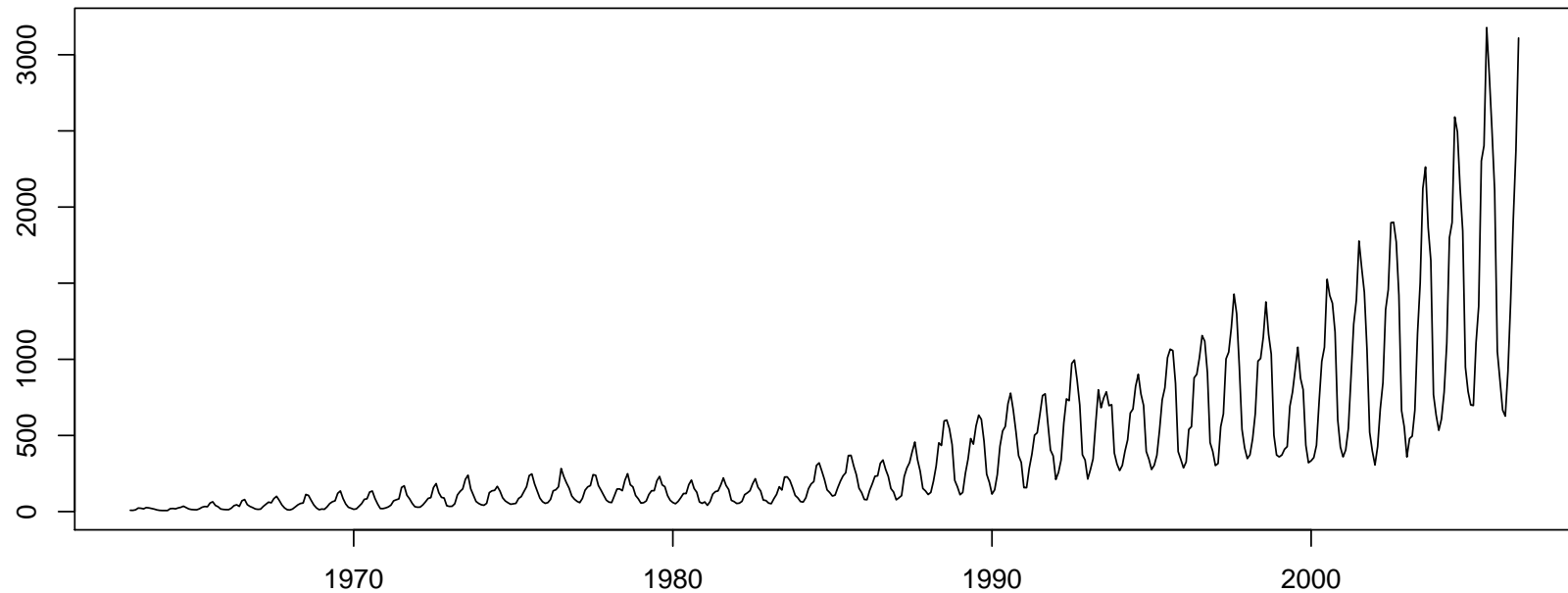
3.7 Example: Crude Oil Inventories

Diagnostics for ARIMA(2,1,0).



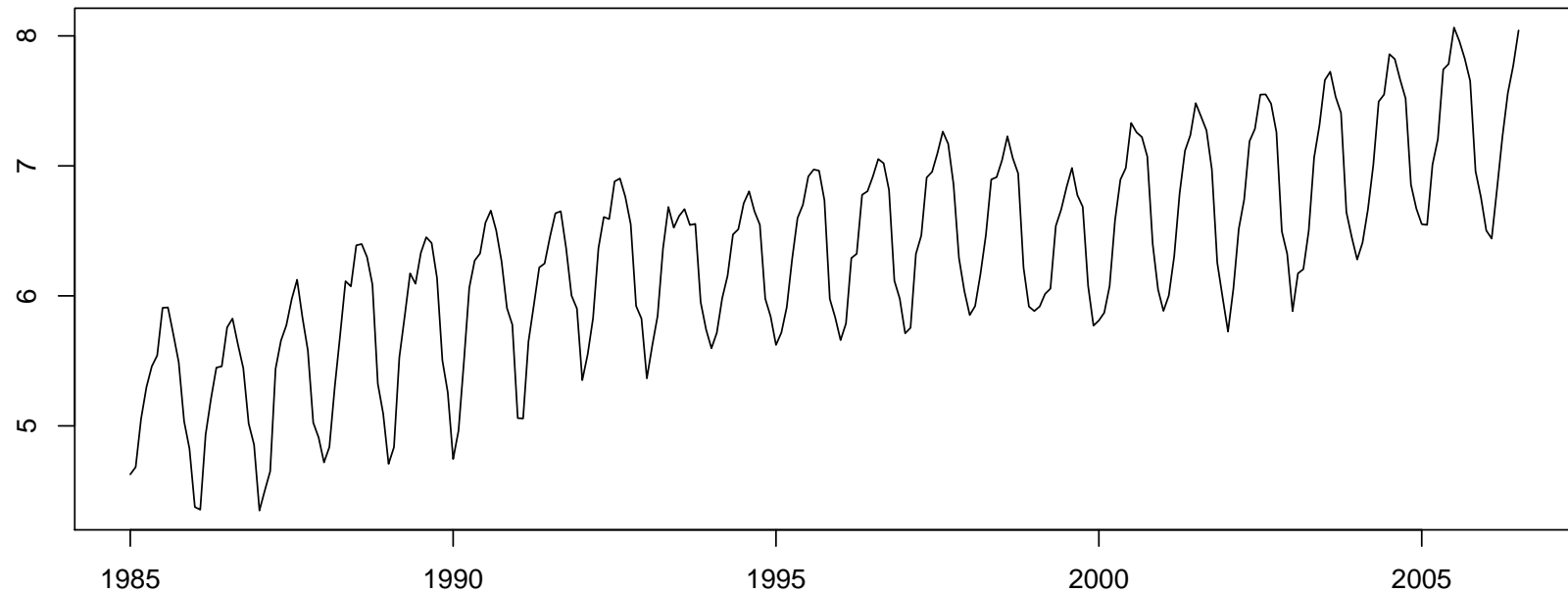
3.8 Example: Tourism in Turkey

Monthly tourist arrivals in Turkey.



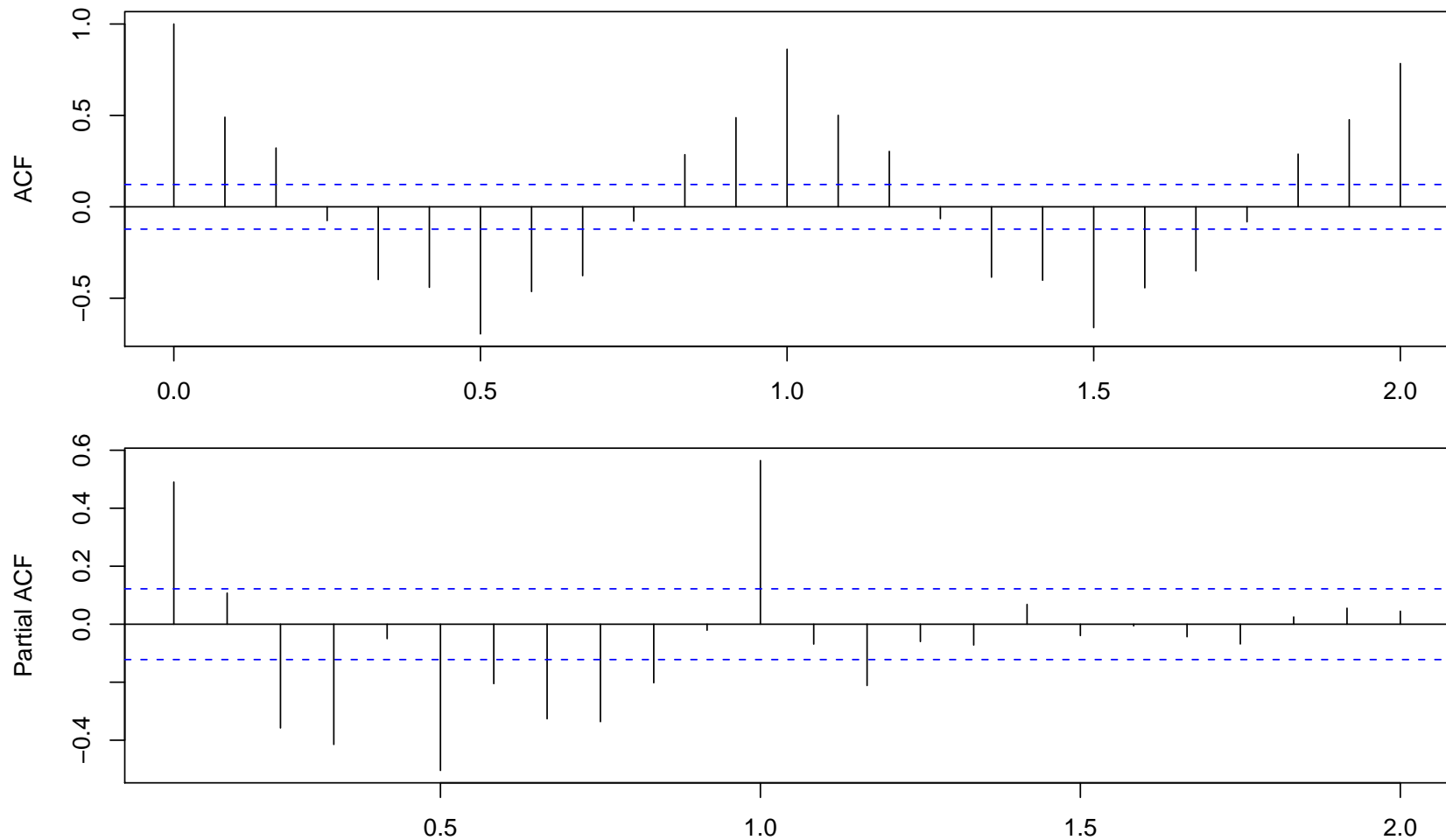
3.8 Example: Tourism in Turkey

Monthly tourist arrivals in Turkey — logged series.



3.8 Example: Tourism in Turkey

Monthly tourist arrivals — acf and pacf of differenced logged series.



3.8 Example: Tourism in Turkey

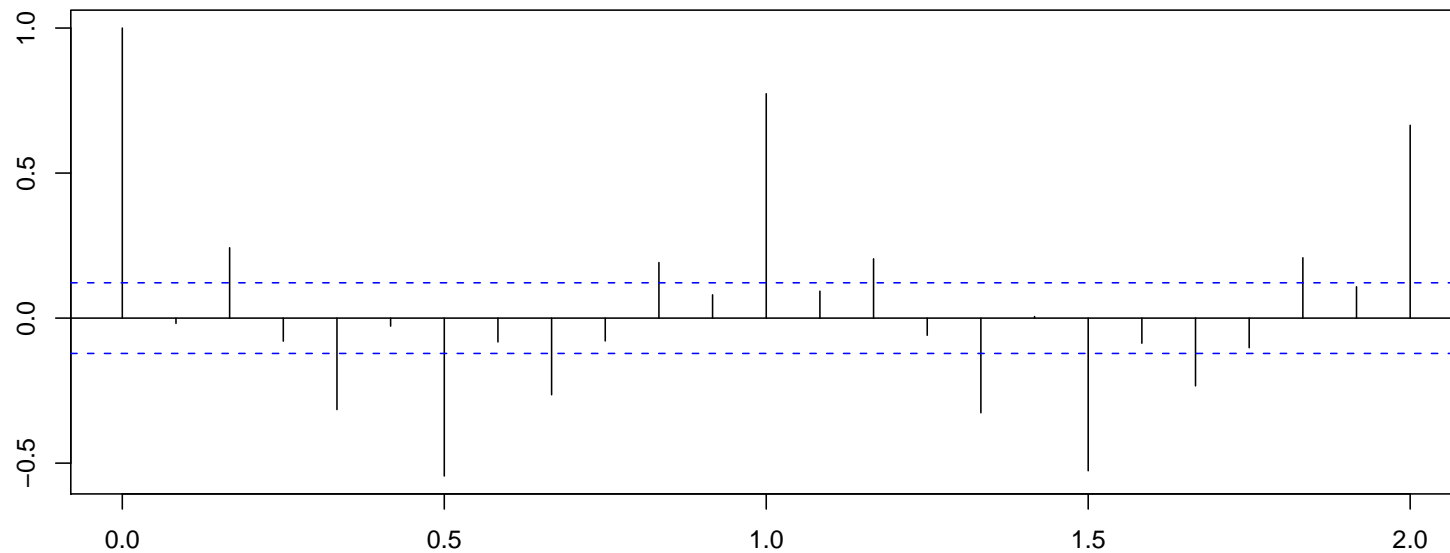
First trial: ARIMA(1,1,1).

```
arima(x = log.tou, order = c(1, 1, 1))
```

Coefficients:

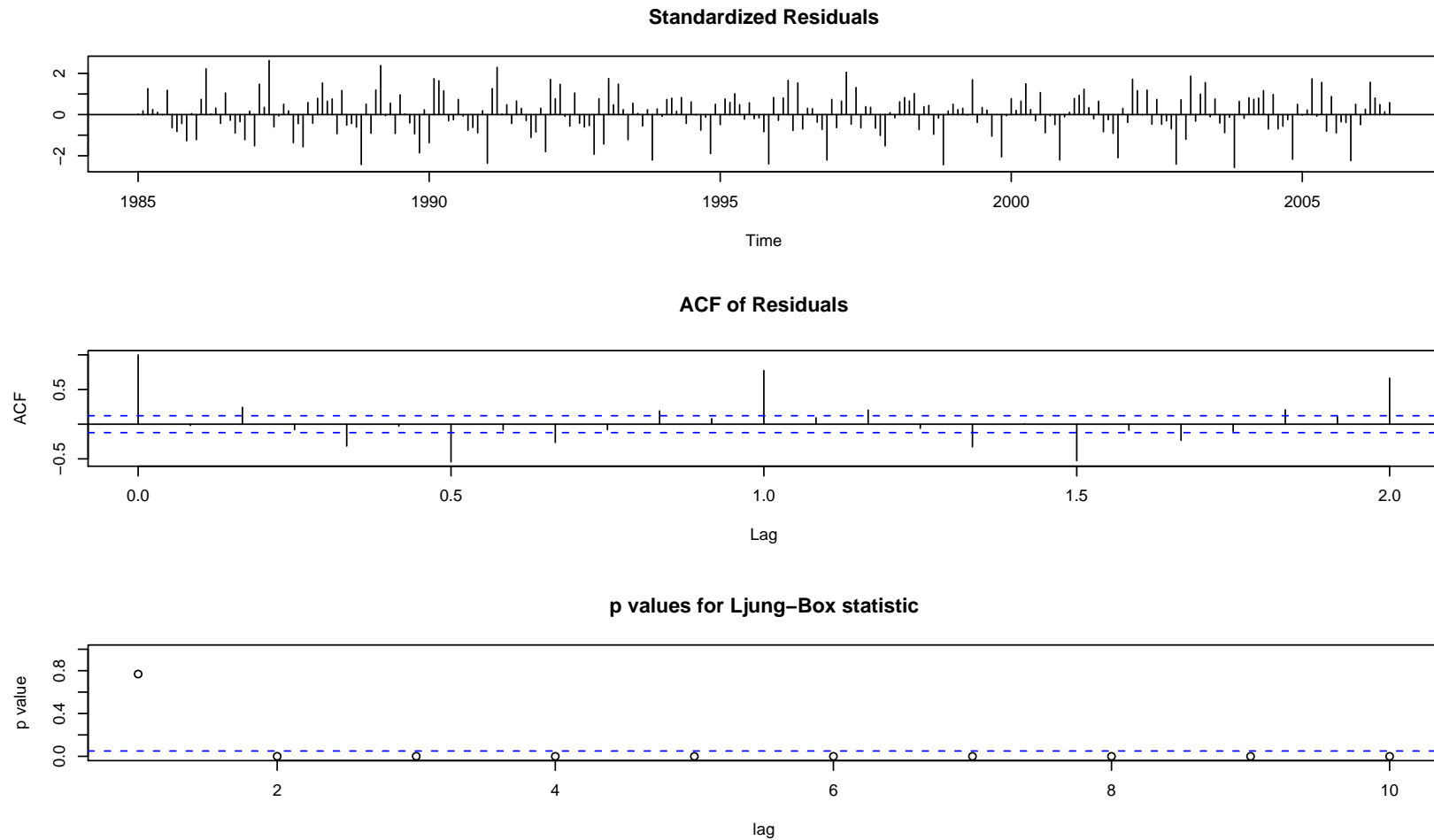
	ar1	ma1
	0.5561	-0.0838
s.e.	0.0748	0.0749

sigma² estimated as 0.07386: log likelihood = -30.1, aic = 66.21



3.8 Example: Tourism in Turkey

First trial: ARIMA(1,1,1).



3.8 Example: Tourism in Turkey

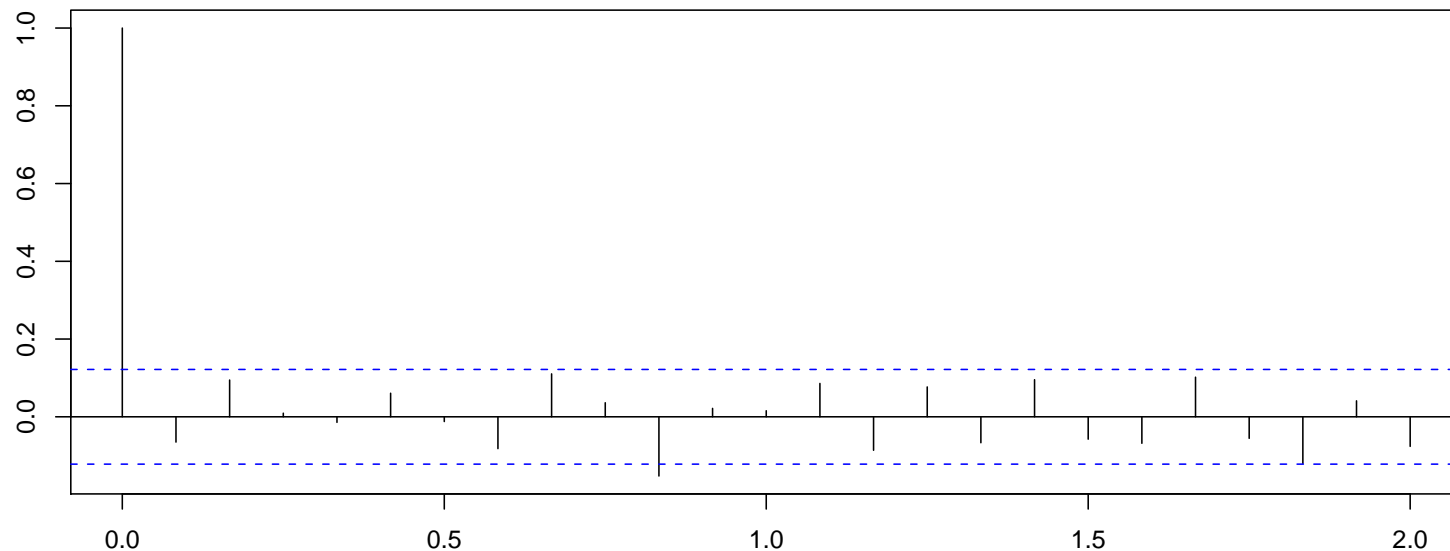
Second trial: SARIMA(1,1,1)×(1,1,1)₁₂.

```
arima(x = log.tou, order = c(1, 1, 1), seasonal = list(order = c(1, 1, 1)))
```

Coefficients:

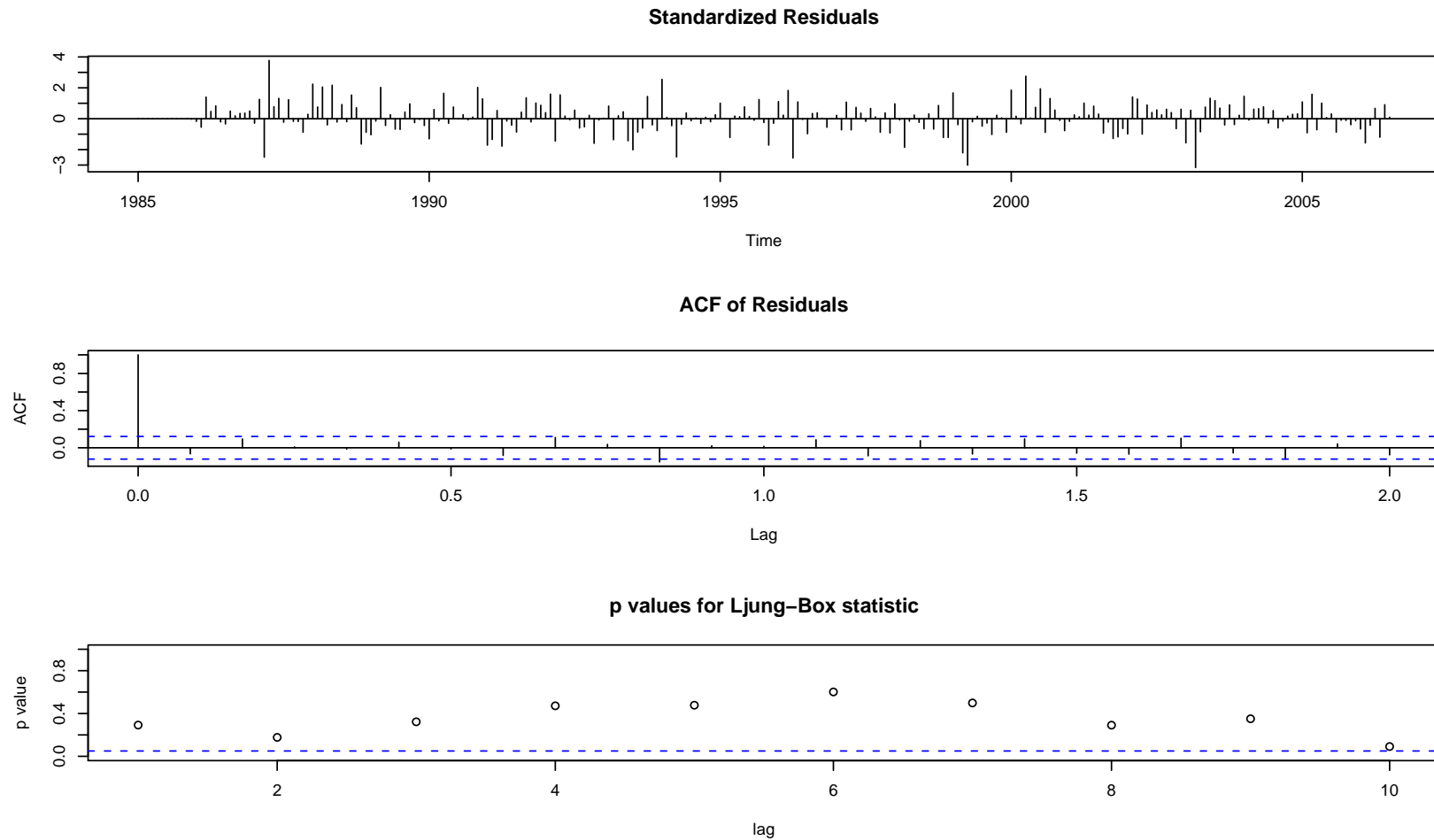
	ar1	ma1	sar1	sma1
	0.6138	-0.8926	0.2803	-0.870
s.e.	0.1535	0.1123	0.0882	0.061

sigma² estimated as 0.01264: log likelihood = 182.53, aic = -355.05



3.8 Example: Tourism in Turkey

Second trial: SARIMA(1,1,1) × (1,1,1)₁₂.



3.8 Example: Tourism in Turkey

The estimated model.

Let X_t = tourist arrivals in month t , and define the differenced series

$$Y_t = (1 - L)(1 - L^{12}) \ln(X_t).$$

The estimated model is:

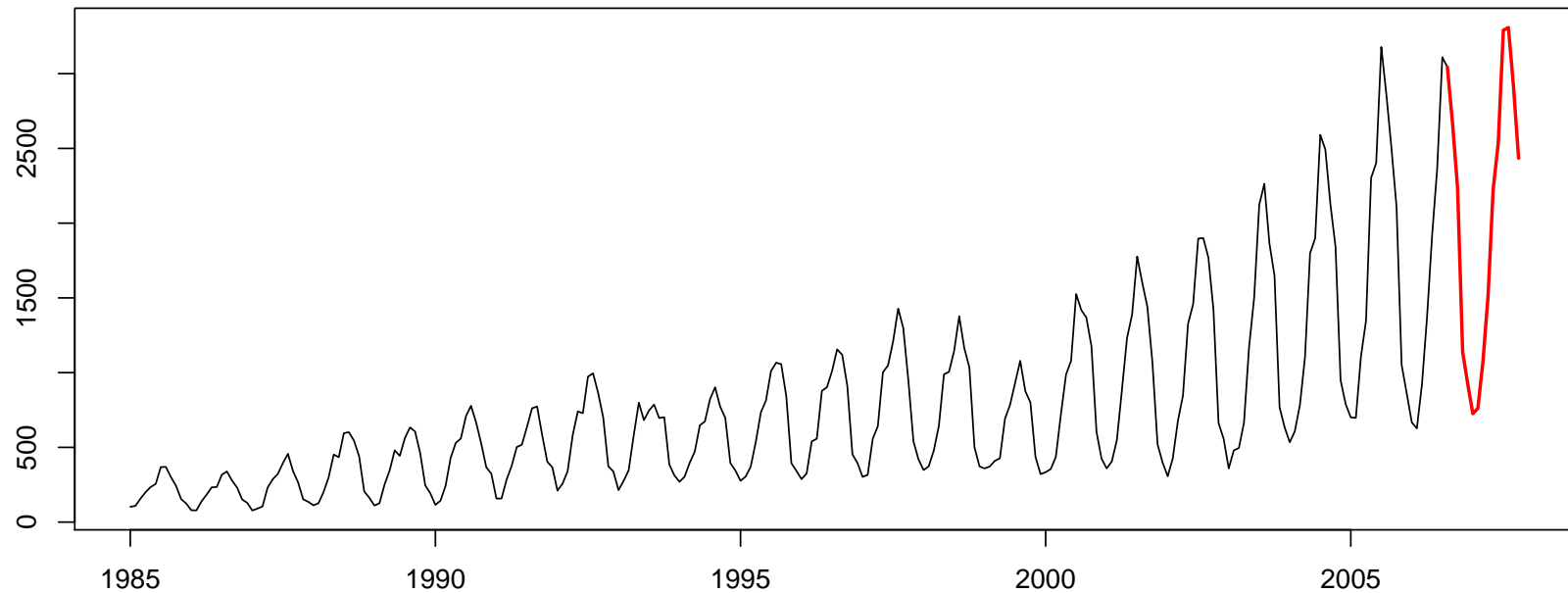
$$(1 - 0.614L)(1 - 0.280L^{12})Y_t = (1 - 0.893L)(1 - 0.870L^{12})\epsilon_t,$$

where (ϵ_t) is Gaussian white noise with variance 0.013.



3.8 Example: Tourism in Turkey

Monthly tourist arrivals in Turkey — observed and predicted.



3.9 Limitations and Outlook

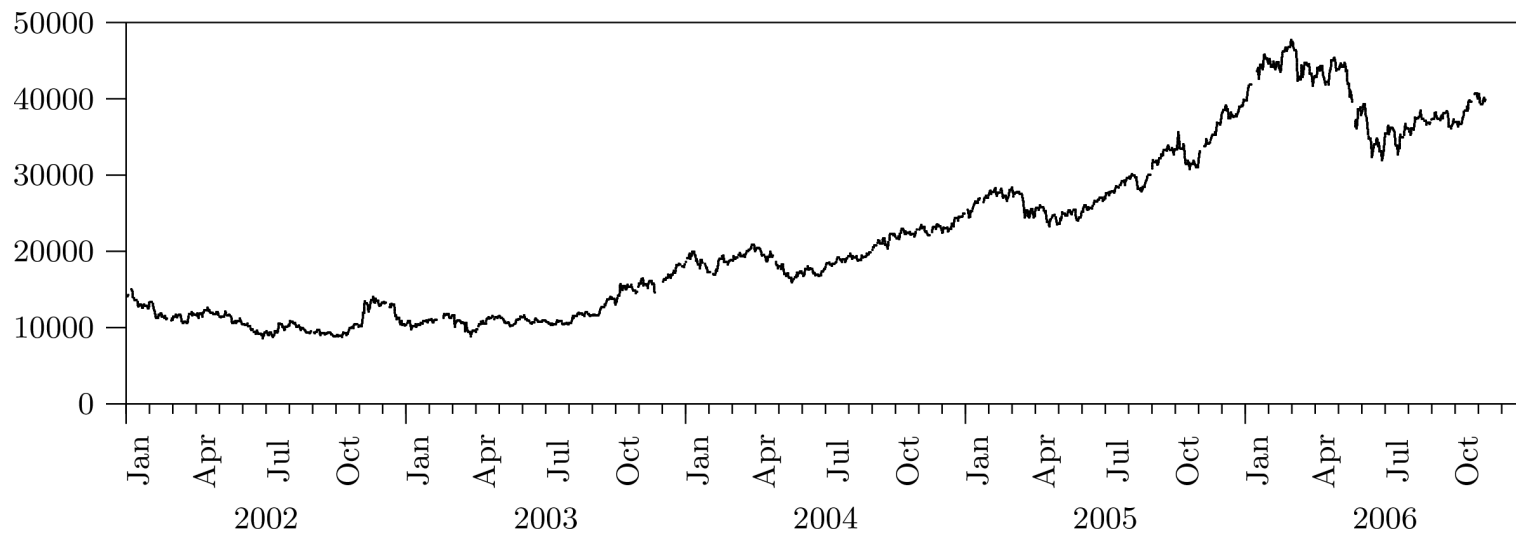
Limitations of ARMA processes in financial modeling.

- ARMA models are conditional *expectation* models.
- ARMA models are NOT conditional *variance* models.
- This means: ARMA models are not really suitable for series which are heteroskedastic.
- GARCH models are conditional *variance* models.
The essential part of a GARCH model is a dynamic specification of the conditional variance.



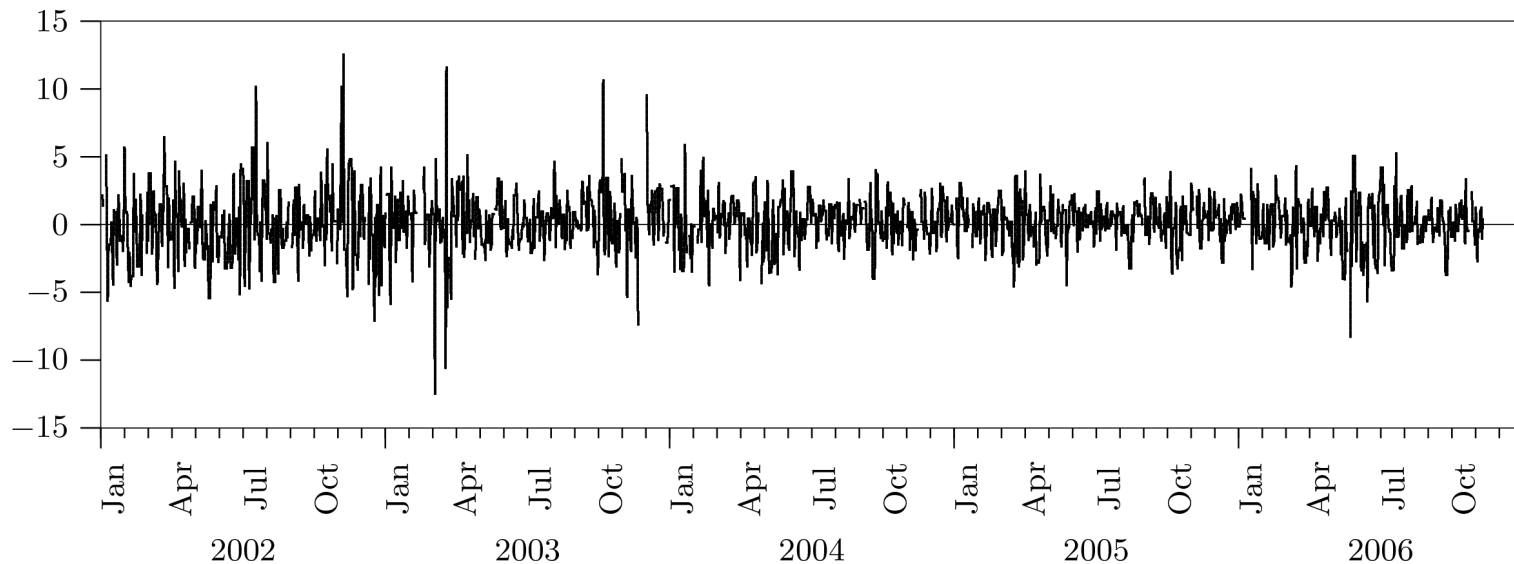
3.9 Limitations and Outlook

Example: İMKB 100 — the level series.



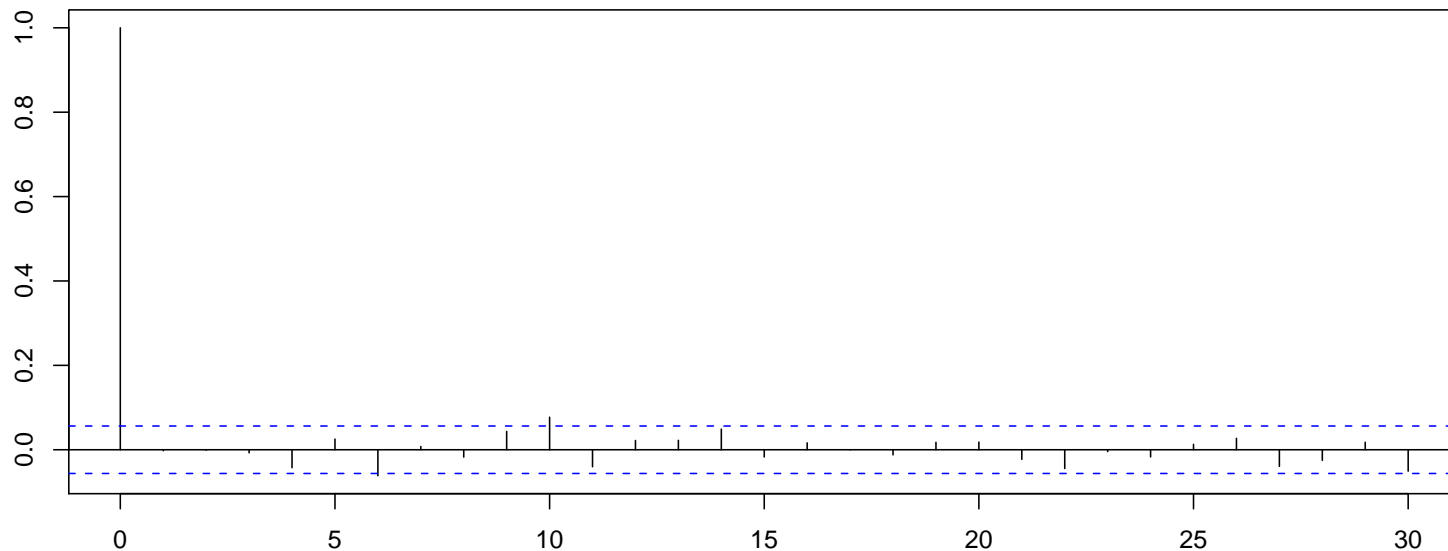
3.9 Limitations and Outlook

Example: IMKB 100 — the return series.



3.9 Limitations and Outlook

Example: IMKB 100 — acf of the return series.



3.9 Limitations and Outlook

Example: IMKB 100 — acf of the squared return series.

