Bus 701: Advanced Statistics

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İSTANBUL BİLGİ ÜNİVERSİTESİ

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- The slides were produced using LaTeX and R (the R project; www.R-project.org) on a Linux system.

- R files used for this course are available upon request.

PART II:

Probability and Stochastic Models

Chapter 5:

Stochastic Models Behind the Observations

5.1 Data and Stochastic Models

Randomness. — There is "randomness" in observed data:

- Drawing another sample will lead to a different selection.
- Many future events cannot be predicted with certainty.

Questions in this context:

- How were the data at hand produced?
- What is behind the data?

Inductive statistics. . .

- . . . is an effort to answer these questions.
- . . . needs probability.

5.1 Data and Stochastic Models

Inductive statistics.

- The paradigm of inductive statistics is:

> **Regard the observations as the outcome of a random experiment, that is, as being produced by a stochastic model.**

- Stochastic model: a mathematical model on the basis of probability.

- The object of research is then the stochastic model, rather than the observations!

5.1 Data and Stochastic Models

Example: Throwing a die once.

This is a random experiment.

- **Before** the die is thrown:

$$X = \text{number which appears}$$

  is a random variable, and $P(X = i) = 1/6$.

- **After** the die is thrown, a probability statement is no longer meaningful! But we can still see the result as being produced by a chance setup.

5.1 Data and Stochastic Models

Example: A public opinion poll.

- Q: "Do you think New Orleans should be rebuilt?"
- Define: $p =$ share of American adults who say "YES"
- For each randomly selected person, we have a random variable:

$$X = \begin{cases} 1 & \text{if the person says "YES"} \\ 0 & \text{if the person says "NO"} \end{cases}$$

- Then, $P(X = 1) = p$ — the share p can be seen as a probability!
- How can we learn about p?

5.1 Data and Stochastic Models

Example: A public opinion poll.

- Q: "Do you think New Orleans should be rebuilt?"

- There is empirical evidence:

  384 out of 609 randomly selected American adults said "YES". (According to CNN, 2005-09-08.)

- We can then estimate p:

$$\hat{p} = \frac{384}{609} = 63\%.$$

- Observe that p and $\hat{p}$ are different objects!

5.1 Data and Stochastic Models

Observations and stochastic models:
analogies and their relation.

OBSERVATIONS

- variables
- observed values
- relative frequency
- empirical distribution

statistical inference →

←

- random experiments
- computer simulations

STOCHASTIC MODEL

- random variables
- their realizations
- probability
- probability distribution

5.1 Data and Stochastic Models

Again: Why do we need stochastic models?

- The entire population is always identified with a stochastic model.

- A random sample allows us to learn about the stochastic model.

- The estimated model represents what we know about the population.

5.2 Probability Calculations

How to obtain a stochastic model. . .

Basic questions:

- Which outcomes are possible?
  Which values can the random variable take on?

- Which probabilities can be assigned to sets of possible outcomes?

5.2 Probability Calculations

Events and probabilities.

- An event is a set of possible outcomes of a random experiment.
  An event has a certain propensity (tendency) to occur.

- This propensity is expressed by a number between 0 and 1, the probability of the event.

- Probabilities must not be assigned totally arbitrarily to events! Certain rules must not be violated.

5.2 Probability Calculations

Kolmogorov's axioms.

i) Every event A has a probability $P(A) \geq 0$.

ii) $P(\Omega) = 1$, where Ω is the set of *all* possible outcomes.

iii) $P(A \cup B) = P(A) + P(B)$ for disjoint events A and B.

From this, further rules can be derived, for example:

$$\begin{aligned} A \subset B \quad &\Rightarrow \quad P(A) \leq P(B) \\ P(\bar{A}) \quad &= \quad 1 - P(A) \\ P(B) \quad &= \quad P(A \cap B) + P(\bar{A} \cap B) \\ P(A \cup B) \quad &= \quad P(A) + P(B) - P(A \cap B) \end{aligned}$$

5.2 Probability Calculations

How can we find the probability of an event?

Two important special cases. . .

- **Laplace experiments:** finite number of outcomes, each with equal probability. Then:

$$P(A) = \frac{\#\text{ outcomes favourable for } A}{\#\text{ possible outcomes}}$$

- **Urn models:** We are now going to look at a simple urn model. . .
  Modifications of the model will carry us very far in Chapters 7, 8 and 9.

5.3 Urn Models

Urn models: an example.

An urn contains 10 balls:

8 balls are red,   2 balls are blue.

Now suppose two balls are randomly drawn from the urn.
We want to find:

$$P(\underbrace{\text{the } 1^{\text{st}} \text{ ball is red}}_{\text{event } A} \text{ and } \underbrace{\text{the } 2^{\text{nd}} \text{ ball is blue}}_{\text{event } B})$$

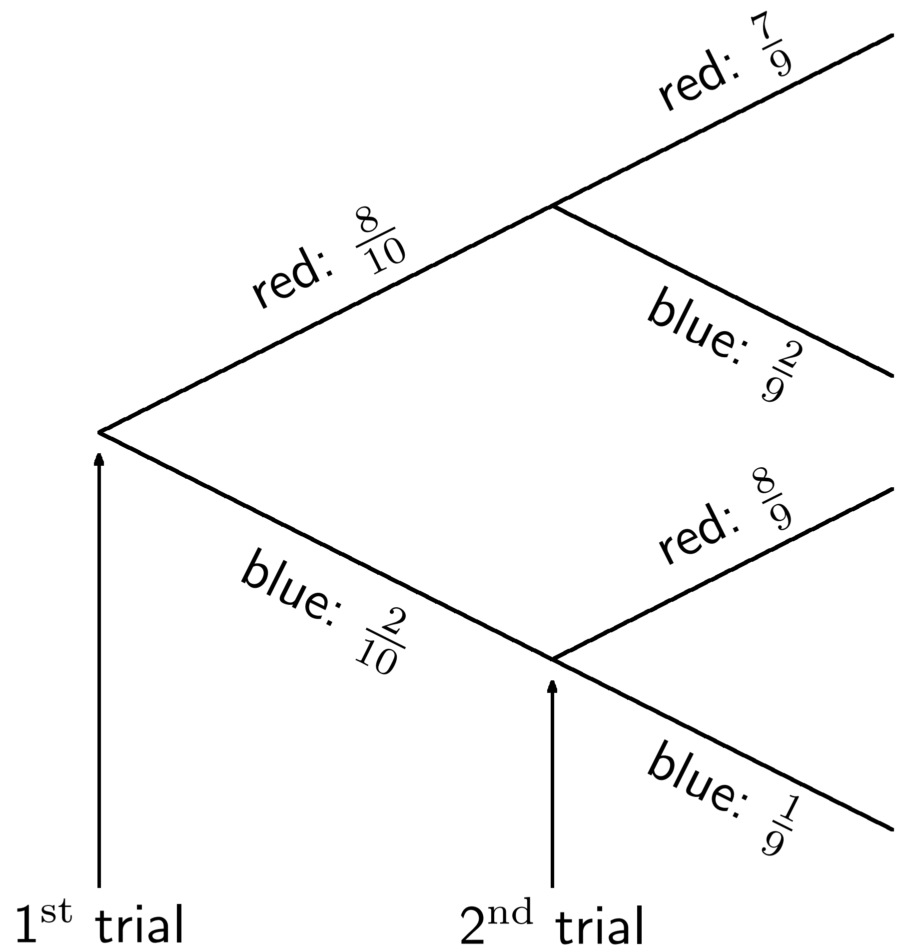
5.3 Urn Models

An urn model.

Drawing without replacement:

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{8}{10} \cdot \frac{2}{9}$$

The events A and B are dependent.



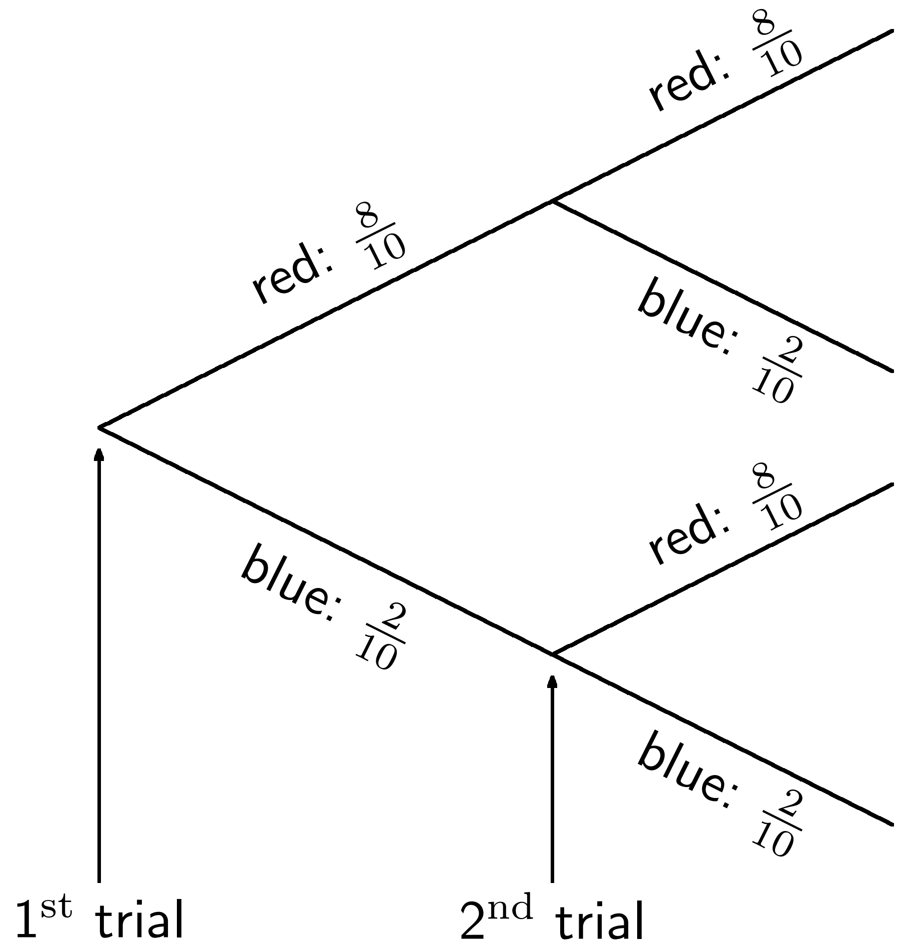
5.3 Urn Models

An urn model.

Drawing with replacement:

$$
\begin{aligned}
P(A \cap B) &= P(A) \cdot P(B|A) \\
&= P(A) \cdot P(B) \\
&= \frac{8}{10} \cdot \frac{2}{10}
\end{aligned}
$$

The events A and B are independent.



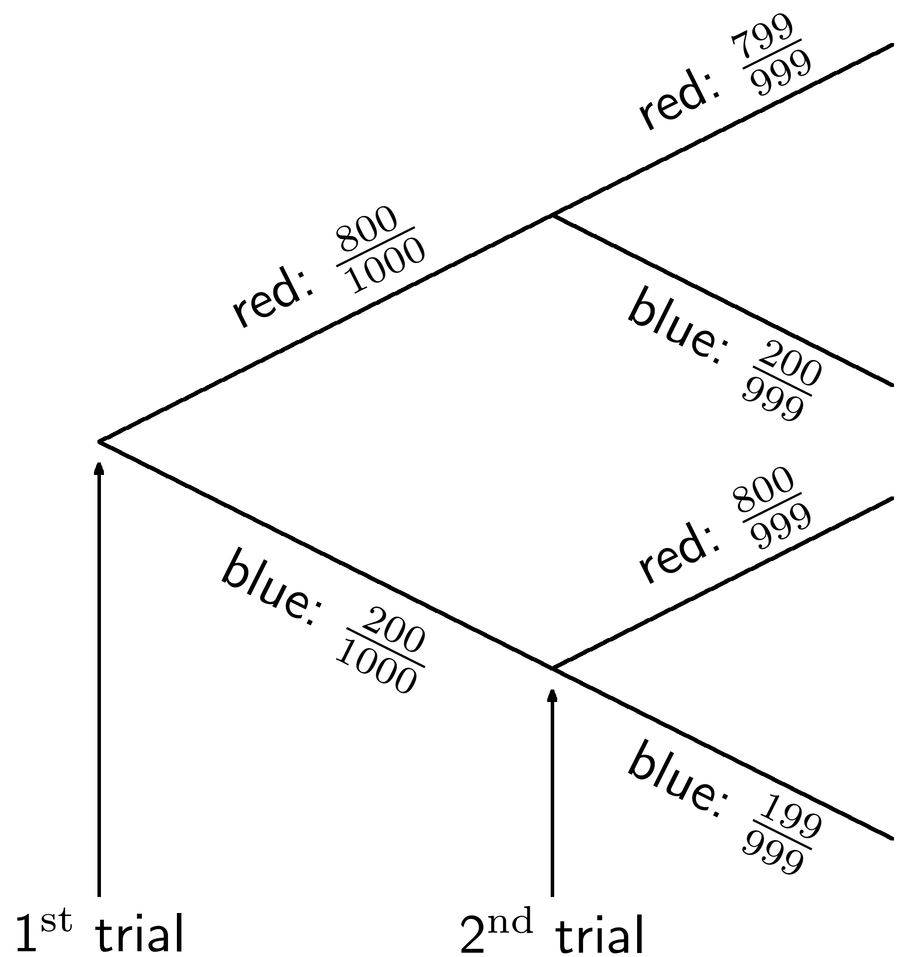
5.3 Urn Models

An urn model.

Drawing without replacement, but large number of balls:

$$
\begin{aligned}
P(A \cap B) &= P(A) \cdot P(B|A) \\
&\approx P(A) \cdot P(B) \\
&= \frac{8}{10} \cdot \frac{2}{10}
\end{aligned}
$$

The events A and B are (almost) independent.



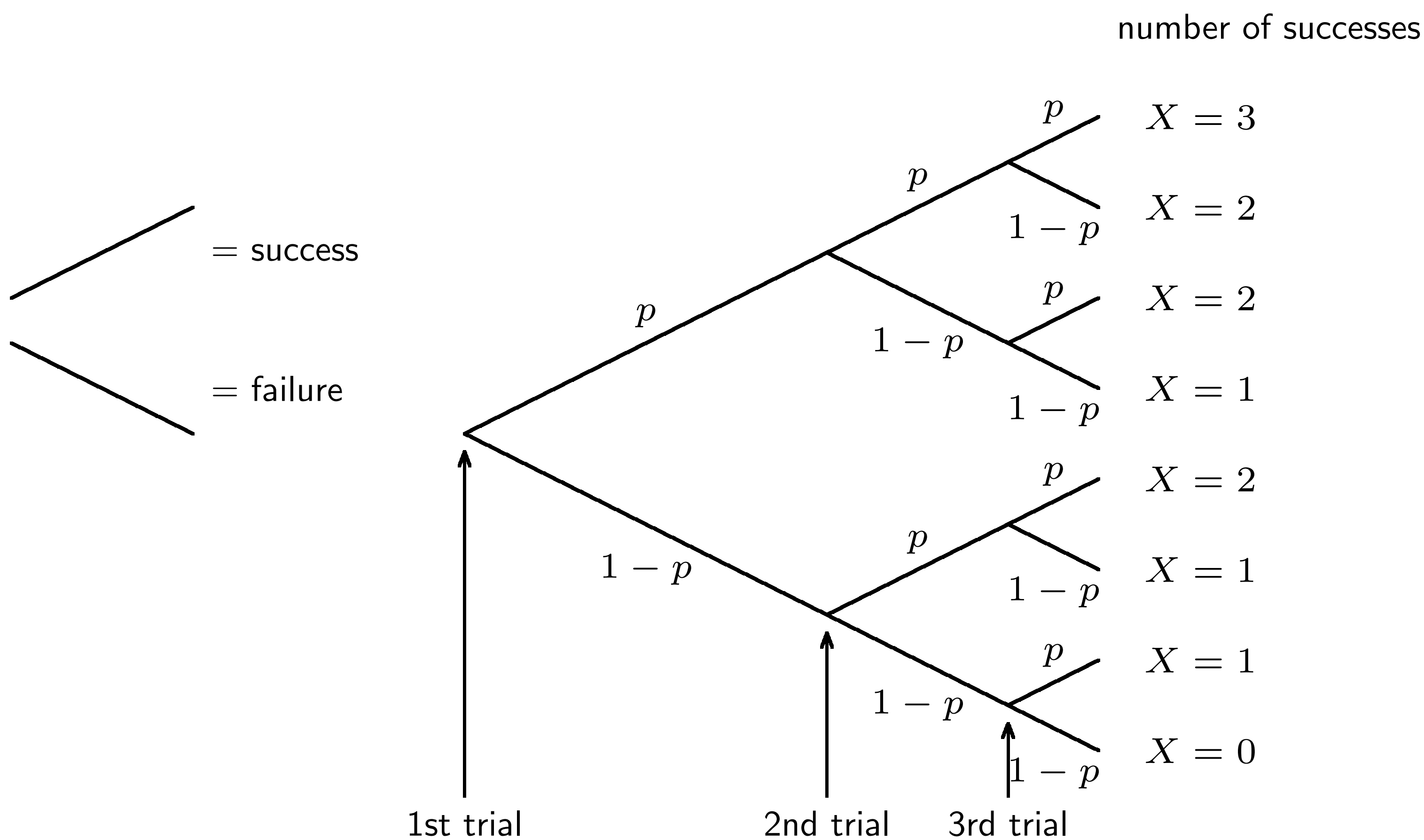
5.3 Urn Models

Consequences for sampling (with a binary variable).

- Sampling with replacement is like complete random sampling.
- Sampling without replacement reduces the uncertainty if the population is small.
- Sampling without replacement from a large population:
  We may act as if the sample was obtained by complete random sampling.

5.3 Urn Models

A setup with three independent trials:



5.3 Urn Models

Bernoulli trials; the binomial distribution. . .

- Combining branches for each event $\{X = i\}$:

$$
\begin{aligned}
P(X=0) &= (1-p)^3 \\
P(X=1) &= 3p(1-p)^2 \\
P(X=2) &= 3p^2(1-p) \\
P(X=3) &= p^3
\end{aligned}
$$

5.3 Urn Models

Bernoulli trials; the binomial distribution. . .

- This can be written more elegantly:

$$P(X = i) = \binom{3}{i} p^i (1-p)^{3-i}, \quad i = 0, \ldots, 3.$$

- This is a special case of the binomial distribution:

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, \ldots, n.$$

5.4 Conditional Probability

Definition of conditional probability.

- Let A be an event such that $P(A) > 0$. Then,

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

  is called the conditional probability of B on condition A.

- With this:

$$\begin{aligned} P(A \cap B) &= P(B|A) \cdot P(A) \\ P(B) &= P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A}) \end{aligned}$$

5.4 Conditional Probability

Examples.

- Rolling a die once. Let $A = \{4,5,6\}, B = \{4,6\}$.

  Then, $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{1/3}{1/2} = \dfrac{2}{3}$.

- What is $P(B)$ in the urn example (drawing without replacement, page 17)?

$$P(B|A) = \frac{2}{9}, \quad P(B|\bar{A}) = \frac{1}{9};$$

therefore,

$$P(B) = \frac{2}{9} \cdot \frac{8}{10} + \frac{1}{9} \cdot \frac{2}{10} = \frac{2}{10}.$$

5.5 The Bayes Theorem

The Bayes theorem and its purpose.

- The Bayes theorem states that

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B|A) \cdot P(A) + P(B|\bar{A}) \cdot P(\bar{A})}$$

- It permits to find the "reverse probability" $P(A|B)$.

- The Bayes theorem is the tool for processing empirical information in a belief-type context.
  (See Chapter 6.)