Bus 274:
Further Statistics For Business

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Chapter 17:

Analysis of Variance

(PART — DRAFT VERSION)

17.1 Introduction

The scope of ANOVA.

- ANOVA is a method to test the hypothesis that the means of several normally distributed populations are equal.

- The question

  Are averages equal?

  can also be asked as:

  Where does the variability come from?

17.1 Introduction

The scope of ANOVA.

- Without further information, we can only analyze the variability in a data set, but we cannot explain where it comes from:

- If additional information is given, part of the variability can be accounted for:

17.1 Introduction

Example: Used cars.

- Is there a price difference between cars of different colours?
- To find out, we have to compare the means of several (say, K) populations:
  - blue cars
  - green cars
  - red cars
  - ...

17.1 Introduction

Example: A course at university.

- A course at university is taught by three lecturers:

  Ayşe, Bürzel, Çağla.

- There are three populations of students.

- Is there a difference between the populations with respect to their average grades?

17.1 Introduction

Example: A bank.

- A bank has three branch offices.
- Are average service times of customers the same in the branch offices?

17.1 Introduction

Example: A bank.

- A bank has three branch offices.
- Managers have classified the bank customers into to types.
- Are there any differences between customer types / branch offices with respect to average service time?
- How many populations are there in this case?

17.2 One-Way ANOVA

The model.

- Compare K populations ("groups", "treatments") with respect to their means.

- Random variables:

$$X_1, X_2, \ldots, X_K, \quad X_i \sim \mathsf{N}(\mu_i, \sigma^2), \quad i = 1, 2, \ldots, K$$

- This can be written as:

$$X_i = \mu + G_i + \epsilon_i, \quad \epsilon_i \sim \mathsf{N}(0, \sigma^2), \quad i = 1, 2, \ldots, K,$$

where $\sum G_i = 0$.

17.2 One-Way ANOVA

The hypotheses.

- Null hypothesis:

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_K$$

$$H_0 : G_1 = G_2 = \ldots = G_K = 0$$

- Alternative hypothesis:

$$H_1 : \text{There are } i_1, i_2,\ i_1 \neq i_2, \text{ with } \mu_{i_1} \neq \mu_{i_2}$$

$$H_1 : \text{There is an } i \text{ with } G_i \neq 0$$

17.2 One-Way ANOVA

The random sample.

- From each group $i = 1, \ldots, K$, a random sample of size n_i is drawn:

$$X_{ij} \sim \mathsf{N}(\mu_i, \sigma^2), \quad i = 1, \ldots, K, \;\; j = 1, \ldots, n_i$$

- This can be written as a "fixed effects model":

$$X_{ij} = \mu + G_i + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathsf{N}(0, \sigma^2)$$

- Taking expectations:

$$E(X_{ij}) = \mu + G_i$$

17.2 One-Way ANOVA

The random sample.

- Arranging the data in a table:

group					
1	$\dots$	i	$\dots$	K	
X_{11}	$\dots$	X_{i1}	$\dots$	X_{K1}	
X_{12}	$\dots$	X_{i2}	$\dots$	X_{K2}	
$\vdots$		$\vdots$		$\vdots$	
X_{1n_1}	$\dots$	X_{in_i}	$\dots$	X_{Kn_K}	
$\bar{X}_1$	$\dots$	$\bar{X}_i$	$\dots$	$\bar{X}_K$	$\bar{X}$

- Sample means:

$$\bar{X}_i = \frac{1}{n_i}\sum_{j=1}^{n_i} X_{ij} \text{ (group } i\text{)}, \quad \bar{X} = \frac{1}{n}\sum_{i=1}^{K}\sum_{j=1}^{n_i} X_{ij} \text{ (overall)}$$

17.2 One-Way ANOVA

The "empirical group effect".

- The empirical analogue to $E(X_{ij}) = \mu + G_i$ is:

$$\bar{X}_i = \bar{X} + (\bar{X}_i - \bar{X})$$

- The term $\bar{X}_i - \bar{X}$ measures the group effect.
- How important is the group effect? —
  We use sums of squares to measure variabilities within and between groups.

17.2 One-Way ANOVA

Sums of squares.

- within groups: $\text{SSW} = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2$, df: $n - K$
- between groups: $\text{SSG} = \sum_{i=1}^{K} n_i (\bar{X}_i - \bar{X})^2$, df: $K - 1$
- total: $\text{SST} = \sum_{i=1}^{K} \sum_{j=1}^{n_i} (X_{ij} - \bar{X})^2$, df: $n - 1$

17.2 One-Way ANOVA

Decomposition of variance.

- Total variability can be decomposed as follows:

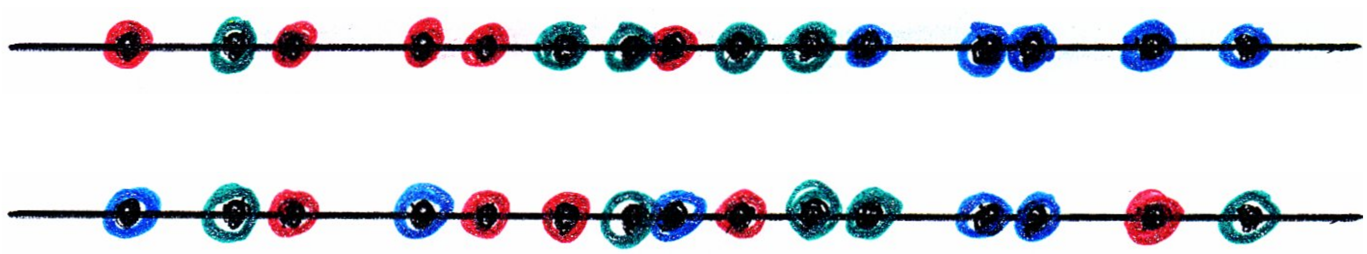
$$
\begin{array}{ccccc}
\text{SST} & = & \text{SSW} & + & \text{SSG} \\
n-1 & = & (n-K) & + & (K-1)
\end{array}
$$

- How can this contribute to testing our hypothesis?

17.2 One-Way ANOVA

Decomposition of variance.

- Consider two situations:



- SST is the same in both cases, but it is decomposed into different components.
- H_0 might be rejected in the first case.

17.2 One-Way ANOVA

Mean sums of squares.

- $\text{MSW} = \frac{\text{SSW}}{n-K} = \frac{1}{n-K}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(X_{ij}-\bar{X}_i)^2$, df: $n-K$
- $\text{MSG} = \frac{\text{SSG}}{K-1} = \frac{1}{K-1}\sum_{i=1}^{K} n_i(\bar{X}_i-\bar{X})^2$, df: $K-1$
- $\text{MST} = \frac{\text{SST}}{n-1} = \frac{1}{n-1}\sum_{i=1}^{K}\sum_{j=1}^{n_i}(X_{ij}-\bar{X})^2$, df: $n-1$

17.2 One-Way ANOVA

Testing the null hypothesis $H_0 : \mu_1 = \ldots = \mu_K$.

- If H_0 is true,

$$F = \frac{\mathsf{MSG}}{\mathsf{MSW}} \sim F_{K-1,n-K}$$

- Critical for H_0: too large values of F.

17.2 One-Way ANOVA

The ANOVA table.

- It contains all relevant measures.
- One-way ANOVA table:

Source of variation	SS	df	MS	F_{calc}	F_{crit}
between groups	SSG	$K-1$	MSG	MSG/MSW	
within groups	SSW	$n-K$	MSW		
total	SST	$n-1$			

17.3 Two-Way ANOVA

Introduction.

- Two-way ANOVA assumes that the population is stratified (cross-classified) in two ways:
  - with respect to K groups,
  - with respect to H blocks.
- Each combination of group and block is represented by a random variable:

$$X_{ij} \sim \mathsf{N}(\mu_{ij}, \sigma^2), \quad i = 1, \ldots, K,\ j = 1, \ldots, H$$

- Example: See handout.

17.3 Two-Way ANOVA

The model.

- Random variables:

$$X_{ij} \sim \mathsf{N}(\mu_{ij}, \sigma^2), \quad i = 1, \ldots, K, \; j = 1, \ldots, H$$

- This can be written as:

$$X_{ij} = \mu + G_i + B_j + I_{ij} + \epsilon_{ij}, \quad \epsilon_{ij} \sim \mathsf{N}(0, \sigma^2)$$

with $\sum_i G_i = \sum_j B_j = \sum_{i,j} I_{ij} = 0$.

- Taking expectations:

$$\mu_{ij} = \mu + G_i + B_j + I_{ij}$$

17.3 Two-Way ANOVA

The meaning of μ, G_i, B_j, and I_{ij}.

- μ is the overall mean.
- G_i reflects differences between groups.
- B_j reflects differences between blocks.
- I_{ij} reflects interactions between groups and blocks.

17.3 Two-Way ANOVA

The null hypotheses.

The goal of two-way ANOVA is to test three hypotheses, using the same data set:

- H_0 : There is no group effect, that is: $G_i = 0$ for all i.
- H_0 : There is no block effect, that is: $B_j = 0$ for all j.
- H_0 : There are no interactions, that is: $I_{ij} = 0$ for all i, j.

17.3 Two-Way ANOVA

Example: Fuel consumption of cars.

- How do driver/car combinations of
  - cars from three car types (the groups) and
  - drivers from three age classes (the blocks)

  perform w.r.t. fuel consumption?

- The following tables show *assumed* values of μ_{ij}, G_i, B_j, and I_{ij}.

- We shall see two situations:
  one with and one without interactions.

17.3 Two-Way ANOVA

Example: Fuel consumption of cars. Situation I:

	car type α	β	γ	block mean	B_j
young	9	10	11	10	1
middle	7	8	9	8	-1
old	8	9	10	9	0
group mean	8	9	10	9	
G_i	-1	0	1		

	car type α	β	γ
young	0	0	0
middle	0	0	0
old	0	0	0

- There are no interactions in this case.
- For example,

$$\begin{array}{ccccccccc} \mu_{\gamma,\text{old}} & = & \mu & + & G_\gamma & + & B_{\text{old}} & + & I_{\gamma,\text{old}} \\ 10 & = & 9 & + & 1 & + & 0 & + & 0 \end{array}$$

17.3 Two-Way ANOVA

Example: Fuel consumption of cars. Situation II:

	car type α	β	γ	block mean	B_j
young	9	10	11	10	1
middle	9	8	7	8	-1
old	6	9	12	9	0
group mean	8	9	10	9	
G_i	-1	0	1		

	car type α	β	γ
young	0	0	0
middle	2	0	-2
old	-2	0	2

- Now, interactions exist.
- For example,

$$\begin{array}{ccccccccc} \mu_{\gamma,\text{old}} & = & \mu & + & G_\gamma & + & B_{\text{old}} & + & I_{\gamma,\text{old}} \\ 12 & = & 9 & + & 1 & + & 0 & + & 2 \end{array}$$

17.3 Two-Way ANOVA

Sample and sample means.

Sample: X_{ijl}, $i = 1, \ldots, K$, $j = 1, \ldots, H$, $l = 1, \ldots, L$

	1	$\ldots$	i	$\ldots$	K	
1	$X_{111}, \ldots, X_{11L}$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\bar{X}_{\cdot 1 \cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	$\ldots$	$\ldots$	$X_{ij1}, \ldots, X_{ijL}$	$\ldots$	$\ldots$	$\bar{X}_{\cdot j \cdot}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
H	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$X_{KH1}, \ldots, X_{KHL}$	$\bar{X}_{\cdot H \cdot}$
	$\bar{X}_{1 \cdot \cdot}$	$\ldots$	$\bar{X}_{i \cdot \cdot}$	$\ldots$	$\bar{X}_{K \cdot \cdot}$	$\bar{X}$

Here,

$$\bar{X}_{i \cdot \cdot} = \frac{1}{HL} \sum_{j,l} X_{ijl}, \quad \bar{X}_{\cdot j \cdot} = \frac{1}{KL} \sum_{i,l} X_{ijl}, \quad \bar{X} = \frac{1}{KHL} \sum_{i,j,l} X_{ijl}.$$

17.3 Two-Way ANOVA

Parameters and their empirical analogues.

parameter	empirical analogue
μ	$\bar{X}$
G_i	$\bar{X}_{i\cdot\cdot} - \bar{X}$
B_j	$\bar{X}_{\cdot j \cdot} - \bar{X}$
μ_{ij}	$\bar{X}_{ij\cdot}$
I_{ij}	$\bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X}$

To understand the last relationship, observe that

$$\begin{array}{ccccccccc} I_{ij} & = & \mu_{ij} & - & \mu & - & G_i & - & B_j \\ \bar{X}_{ij\cdot} - \bar{X}_{i\cdot\cdot} - \bar{X}_{\cdot j\cdot} + \bar{X} & = & \bar{X}_{ij\cdot} & - & \bar{X} & - & (\bar{X}_{i\cdot\cdot} - \bar{X}) & - & (\bar{X}_{\cdot j\cdot} - \bar{X}), \end{array}$$

because of $\mu_{ij} = \mu + G_i + B_j + I_{ij}$.

17.3 Two-Way ANOVA

Sums of squares.

between groups: $\text{SSG} = HL \sum_{i=1}^{K} (\bar{X}_{i..} - \bar{X})^2$

between blocks: $\text{SSB} = KL \sum_{j=1}^{H} (\bar{X}_{.j.} - \bar{X})^2$

interaction: $\text{SSI} = L \sum_{i=1}^{K} \sum_{j=1}^{H} (\bar{X}_{ij.} - \bar{X}_{i..} - \bar{X}_{.j.} + \bar{X})^2$

error: $\text{SSE} = \sum_{i=1}^{K} \sum_{j=1}^{H} \sum_{l=1}^{L} (X_{ijl} - \bar{X}_{ij.})^2$

total: $\text{SST} = \sum_{i=1}^{K} \sum_{j=1}^{H} \sum_{l=1}^{L} (X_{ijl} - \bar{X})^2$

17.3 Two-Way ANOVA

The ANOVA table.

Two-way ANOVA table:

source of variation	sum of squares	df	mean squares	F_{obs}	F_{crit}
groups	SSG	$K-1$	MSG	$\frac{\text{MSG}}{\text{MSE}}$	$F_{1-\alpha;K-1,KH(L-1)}$
blocks	SSB	$H-1$	MSB	$\frac{\text{MSB}}{\text{MSE}}$	$F_{1-\alpha;H-1,KH(L-1)}$
interaction	SSI	$(K-1)(H-1)$	MSI	$\frac{\text{MSI}}{\text{MSE}}$	$F_{1-\alpha;(K-1)(H-1),KH(L-1)}$
error	SSE	$KH(L-1)$	MSE		
total	SST	$KHL-1$			