Bus 274:
Further Statistics For Business

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 İSTANBUL BİLGİ ÜNİVERSİTESİ

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- R files used for this course are available upon request.

Chapter 11:

Parameter Tests for One Population

11.1 Introduction

The context.

- Hypothesis testing belongs to inductive statistics.
- Goal of inductive statistics: Make statements about a population on the basis of sample data.
- The population is characterized by a stochastic model; simplest case:
  by a random variable and its distribution.

11.1 Introduction

An example from quality control.

- A glass manufacturer produces bottles for carbonated beverages.
- Important: The bottles should be strong enough.
- Of course we should try to make sure that *each* bottle satisfies the standard.
- But that's not the whole story. . .
- We should focus on monitoring the production process, rather than on particular outcomes.

11.1 Introduction

An example from quality control.

- The production process is characterized by a random variable and its probability distribution:

$$X = \text{internal pressure strength of a randomly selected bottle (in psi)}$$

- The production manager knows from past experience:

$$X \sim \mathsf{N}(\mu, 10^2)$$

- The bottler requires that μ should be at least 175.

11.1 Introduction

An example from quality control.

- That is, we should test

  $H_0 : \mu \le \mu_0 = 175$ (or $H_0 : \mu = \mu_0 = 175$) against
  $H_1 : \mu > \mu_0 = 175.$

- If H_0 can be rejected against H_1, we may say:
  - Average internal pressure strength was found to be significantly larger than 175 psi.
  - There is empirical evidence that average internal pressure strength is larger than 175 psi.

11.1 Introduction

An example from quality control.

- In order to test H_0, we need a sample from the production process: $X_1, \ldots, X_n \sim \mathsf{N}(\mu, 10^2)$; iid

- Test statistic:

$$T = \frac{\bar{X} - \mu_0}{\frac{10}{\sqrt{n}}} \qquad \text{with} \quad \mu_0 = 175$$

- If $H_0 : \mu = \mu_0$ is true, $T \sim \mathsf{N}(0, 1)$.

- Critical for H_0: large values of T.

- With $\alpha = 5\%$, the critical region is $[1.645, \infty)$.

11.1 Introduction

An example from quality control.

- Now suppose we have a sample of 25 bottles, with $\bar{x} = 182$.
- Here,

$$T = \frac{182 - 175}{\frac{10}{\sqrt{25}}} = 3.5 > 1.645.$$

- This means: $H_0 : \mu \leq 175$ is rejected against $H_1 : \mu > 175$.
- The p-value (prob-value) of H_0 against H_1 is

$$P_{\mu_0}(T > 3.5) = 0.00023 = 0.023\%.$$

- How can this be formulated in words?

11.1 Introduction

An example from quality control — further questions.

- What does this mean for the production process?
- Should we re-adjust the production process?
- Could we have made a type I or a type II error?
- Maybe we should draw a larger sample?
- Questions like these (depending on the context, of course) should always be asked after conducting a significance test!

11.1 Introduction

The general situation.

- X: our variable of interest
- $X_1, X_2, \ldots, X_n$: random sample of X
- $x_1, x_2, \ldots, x_n$: realizations of $X_1, X_2, \ldots, X_n$
- The distribution of X depends on a parameter θ.
  (Or θ is the expectation of X.)
- The parameter θ is unknown; we want to test a hypothesis concerning θ.

11.1 Introduction

Outlook on Chapter 11.

- 11.2 Normal Distribution: Testing for μ
  σ^2 known / σ^2 unknown
- 11.3 Approximate Test For the Mean
  No distributional assumption, but the sample needs to be large.
- 11.4 Testing For A Share
  Share or unknown success probability.
- 11.5 Normal Distribution: Testing for σ^2

11.2 Normal Distribution: Testing For μ

Testing for the mean when the variance is known.

- Test problem: $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$
- Test statistic:
$$T = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}}$$
  If H_0 is true, $T \sim \mathrm{N}(0, 1)$.
- Critical for H_0: too small and too large values of T.
- For the one-sided test problem $H_0 : \mu \leq \mu_0$, $H_1 : \mu > \mu_0$, too large values of T are critical.

11.2 Normal Distribution: Testing For μ

Example: Annual water consumption.

- A public utility company assumes that annual water consumption (cubic meters) of households with young families with children is normally distributed with

  mean 135 and standard deviation 55.

- A new residential area was constructed. —
  Is its water consumption "normal"?

- A random sample of 25 households had an average consumption of 127 cubic meters.

11.2 Normal Distribution: Testing For μ

Example: Annual water consumption.

- Is the average water consumption of households in the new area different from 135?
- Define

$$X = \text{annual water consumption of a household in the new area.}$$

- We assume: $X \sim \mathsf{N}(\mu, 55^2)$ and test

$H_0 : \mu = \mu_0 = 135$ against $H_1 : \mu \neq \mu_0 = 135$.

11.2 Normal Distribution: Testing For μ

Example: Annual water consumption.

- The test statistic is

$$T = \frac{\bar{X} - \mu_0}{\frac{\sigma}{\sqrt{n}}} = \frac{127 - 135}{\frac{55}{\sqrt{25}}} = -8/11;$$

  the p-value is:

$$P_{\mu_0}(T < -8/11 \text{ or } T > +8/11) = 0.47$$

- H_0 is not rejected.
- There is no evidence that households in the new area have an unusually low or high water consumption.

11.2 Normal Distribution: Testing For μ

Example: Annual water consumption.

- Observe here the strength of inductive statistics.
- Households in the new area consume less:

general household average:	135
new residential area household average:	127

- But do they systematically consume less?
- We have decided: The difference $127 - 135$ can be explained by random variation. It does not indicate a difference between population means.

11.2 Normal Distribution: Testing For μ

Testing for the mean when the variance is unknown.

- Test problem: $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0$
- Test statistic:

$$T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$

  If H_0 is true, $T \sim t_{n-1}$.

- Critical for H_0: too small and too large values of T.
- For the one-sided test problem $H_0 : \mu \leq \mu_0$, $H_1 : \mu > \mu_0$, too large values of T are critical.

11.2 Normal Distribution: Testing For μ

Example: Packed fruit.

- A fruit packer fills strawberries into packages labeled "500 grams".
- Is there evidence that the packer systematically underfills packages?
- A sample of 15 packages had weights

  490, 488, 491, 494, 473, 475, 510, 480, 503, 481, 497, 501, 480, 509, 487.

- Arithmetic mean: $\bar{x} = 490.6$; standard deviation: $s = 11.71$.

11.2 Normal Distribution: Testing For μ

Example: Packed fruit.

- Let $X =$ weight of fruit in a package.
- We assume: $X \sim \mathrm{N}(\mu, \sigma^2)$, where μ and σ^2 are unknown.
- To be tested:

$$H_0 : \mu = 500 \text{ (or } H_0 : \mu \geq 500\text{) against } H_1 : \mu < 500.$$

- Test statistic:

$$T = \frac{\bar{x} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{490.6 - 500}{\frac{11.71}{\sqrt{15}}} = -3.1091.$$

- The p-value is $P_{\mu_0}(T \leq -3.1091) = 0.0038$.

11.2 Normal Distribution: Testing For μ

Example: Packed fruit.

- What is the critical region for T?
- How can the p-value be explained in words?
- What is our decision with regard to H_0 and H_1?
- Which error may we have made?
- How can we express our findings concerning the packer's filling policy in words?

11.3 Approximate Test For the Mean

Testing for the mean, large sample.

- Test problem: $H_0 : \mu = \mu_0$, $H_1 : \mu \neq \mu_0 \quad (\mu = E(X))$
- Test statistic:
$$T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}}$$
If H_0 is true, $T \sim \mathrm{N}(0, 1)$ approximately for large n.
- Critical for H_0: too small and too large values of T.
- For the one-sided test problem $H_0 : \mu \leq \mu_0$, $H_1 : \mu > \mu_0$, too large values of T are critical.

11.3 Approximate Test For the Mean

Example: Customer expenditure at a supermarket.

- All supermarkets of a certain chain have average customer expenditure €16.15.
- Site analysis, one particular supermarket: Does it deviate from this average?
- Define X = expenditure of a customer.
- To be tested:

$$H_0 : E(X) = \mu_0 = 16.15 \text{ against } H_1 : E(X) \neq \mu_0 = 16.15.$$

11.3 Approximate Test For the Mean

Example: Customer expenditure at a supermarket.

- A sample of 508 customers had $\bar{x} = 15.43$, $s^2 = 166.96$.
- Test statistic:

$$T = \frac{\bar{X} - \mu_0}{\frac{s}{\sqrt{n}}} = \frac{15.43 - 16.15}{\sqrt{\frac{166.96}{508}}} = -1.26$$

- The p-value is $P_{\mu_0}(T \leq -1.26 \text{ or } T \geq 1.26) = 0.208$.
- There is no evidence for significant deviation of average expenditure in this particular supermarket from supermarkets of this chain in general.

11.4 Testing For a Share

Approximate test for a success probability.

- Test problem: $H_0 : p = p_0$, $H_1 : p \neq p_0$
- Test statistic:
$$T = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$$
If H_0 is true, $T \sim \mathsf{N}(0, 1)$ approximately, if n is large.
- Critical for H_0: too small and too large values of T.
- For the one-sided test problem $H_0 : p \leq p_0$, $H_1 : p > p_0$, too large values of T are critical.

11.4 Testing For a Share

Example: Support for an idea.

- A politician claims that a majority of the populace support his strategy to reduce unemployment.
- In a random sample of 1000, 587 said they'd support the strategy.
- Is it justified to say the strategy is supported by a $majority$?
- Define $p =$ share of those in favour of the strategy.
- To be tested: $H_0 : p \leq p_0 = 0.5$ against $H_1 : p > p_0 = 0.5$.

11.4 Testing For a Share

Example: Support for an idea.

- To be tested: $H_0 : p \leq p_0 = 0.5$ against $H_1 : p > p_0 = 0.5$.
- Test statistic:

$$T = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.587 - 0.5}{\sqrt{\frac{0.5^2}{1000}}} = 5.50.$$

- Critical for H_0: too large values of T.
- The critical region is $[1.645, \infty)$ (with $\alpha = 5\%$).
- The p-value is $P_{p_0}(T > 5.50) = P_{p_0}(\hat{p} > 0.587) = 1.9\text{e}-08$.

11.4 Testing For a Share

Example: Ups and downs in the Dow Jones Industrial Average.

	date	close quote	return	indicator
1	2006-01-04	10880.15	0.302%	1
2	2006-01-05	10882.15	0.018%	1
3	2006-01-06	10959.31	0.709%	1
4	2006-01-09	11011.90	0.480%	1
5	2006-01-10	11011.58	−0.003%	0
⋮	⋮	⋮	⋮	⋮
248	2006-12-27	12510.57	0.830%	1
249	2006-12-28	12501.52	−0.072%	0
250	2006-12-29	12463.15	−0.307%	0

11.4 Testing For a Share

Example: Ups and downs in the Dow Jones Industrial Average.

- Among the 250 days in 2006, there were
  - 135 days with an up move,
  - 115 days with a down move.
- Estimated probability of an up move: $\hat{p} = 135/250 = 0.54$
- Does this imply a significant deviation from a supposed 50% chance of an up move?
- Test statistic: $T = 1.26$, p-value: 0.21

11.4 Testing For a Share

Example: Ups and downs in the Dow Jones Industrial Average.

A more sophisticated method to explore a financial market should:

- test a more powerful hypothesis, e.g.:

  The sequence 1,1,1,1,0,. . . is like one created by coin tossing.

- take the magnitude of returns into consideration.
- adopt a *dynamic* stochastic model.

11.5 Normal Distribution: Testing For σ^2

Testing for the variance when the mean μ is unknown.

- Test problem: $H_0 : \sigma^2 = \sigma_0^2$, $H_1 : \sigma^2 \neq \sigma_0^2$
- Test statistic:

$$T = \frac{(n-1)s^2}{\sigma_0^2} = \frac{1}{\sigma_0^2} \sum_{i=1}^{n} (X_i - \bar{X})^2$$

  If H_0 is true, $T \sim \chi^2_{n-1}$.

- Critical for H_0: too small and too large values of T.
- For the one-sided test problem $H_0 : \sigma^2 \leq \sigma_0^2$, $H_1 : \sigma^2 > \sigma_0^2$, too large values of T are critical.

11.5 Normal Distribution: Testing For σ^2

Example: The variability of response times.

- A computerized inquiry system provides information on the availability of air tickets.
- As long as the system is running smoothly, response times are known to be normally distributed with mean 0.6 seconds and standard deviation 0.14 seconds.
- The variance is an important performance parameter.

11.5 Normal Distribution: Testing For σ^2

Example: The variability of response times.

- Samples of response times are frequently drawn to monitor the variance.
- One night, a sample of size 384 was drawn.
- It had $\bar{x} = 0.613$ and $s = 0.1085$.
- Required: Test if the observed variance is in line with the assumption
$$X \sim \mathsf{N}(0.6, 0.14^2).$$

11.5 Normal Distribution: Testing For σ^2

Example: The variance of response time.

- Test problem: $H_0 : \sigma^2 = \sigma_0^2 = 0.14^2$, $H_1 : \sigma^2 \neq \sigma_0^2$
- Test statistic:

$$T = \frac{(n-1)s^2}{\sigma_0^2} = \frac{383 \cdot 0.1085^2}{0.14^2} = 230.0$$

- We need the 2.5% and 97.5% quantiles of the χ^2 distribution with 383 degrees of freedom:

$$\chi^2_{383;0.025} = 330.7; \quad \chi^2_{383;0.975} = 439.1.$$

- Critical region: $[0, 330.7) \cup (439.1, \infty)$.
- H_0 is rejected.