

Bus 273: Statistical Analysis For Business

Harald Schmidbauer



About These Slides

- The present slides are not self-contained; they need to be explained and discussed. This will be done in the lectures.
- Even though being a “work in progress” and subject to revision, the slides constitute copyrighted material.
If you want to reproduce or copy anything from the slides, please ask:

Harald Schmidbauer **harald** at **hs-stat** dot **com**
Angi Rösch **angi** at **angi-stat** dot **com**

- The slides were produced using \LaTeX and R (the R project; website: www.R-project.org) on a GNU/Linux system.
- R files used for this course are available upon request.



Chapter 7: Continuous Probability Distributions



7.1 Basics

Continuous random variables and continuous distributions.

- A random variable is called continuous if it can take on any value in a certain interval.
- The distribution of a continuous random variable is called a continuous distribution.

Examples:

- X = daily return on DAX
- X = waiting time of a customer at a call center until an incoming call is answered
- X = impurity of a chemical produced by a company
-



7.1 Basics

Computing probabilities.

- For a continuous random variable X ,

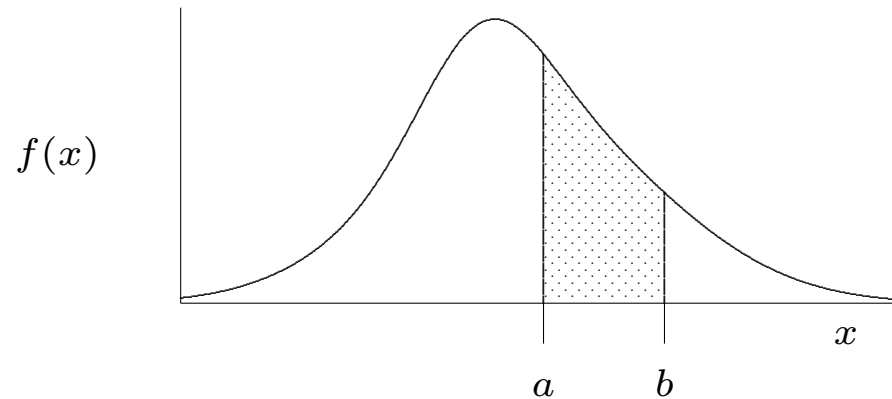
$$P(X = x) = 0 \quad \text{for all } x \in \mathbb{R}.$$

- For example, let $X =$ body-height of a randomly selected person. Then, $P(X = 172) = P(X = 172.0000000) = 0!$
- Therefore, the concept of probability function doesn't work for a continuous random variable.
- A density is needed!



7.1 Basics

The density.



- Probabilities are areas below the density.
- For example,

$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

- A continuous probability distribution is given by these terms.



7.1 Basics

The distribution function.

- The distribution function of X is defined as

$$x \mapsto F(x) = P(X \leq x) = \int_{-\infty}^x f(\xi) d\xi.$$

- With it, $P(a < X \leq b) = F(b) - F(a)$.
- As before, we can
 - display the distribution (here: the density),
 - compute location and variation measures.



7.1 Basics

The expectation: a location measure.

- For a continuous random variable X ,

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx.$$

- Same principle as in Chapter 6, with
 - $f(x)dx$ instead of $p_i = P(X = i)$,
 - \int instead of \sum .



7.1 Basics

The variance: a variation measure.

- For a discrete random variable X ,

$$\begin{aligned}\text{var}(X) &= \mathbf{E} [(X - \mathbf{E}(X))^2] = \mathbf{E}(X^2) - \mathbf{E}^2(X) \\ &= \int_{-\infty}^{\infty} (x - \mathbf{E}(X))^2 \cdot f(x) dx.\end{aligned}$$

- Same principle as in Chapter 6, with
 - $f(x)dx$ instead of $p_i = P(X = i)$,
 - \int instead of \sum .



7.2 The Normal Distribution

Definition.

- A random variable X with density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}, \quad \mu \in \mathbb{R}, \quad \sigma^2 > 0,$$

is said to be normally distributed with parameters μ and σ^2 .

- In symbols: $X \sim N(\mu, \sigma^2)$



7.2 The Normal Distribution

This picture shows the density.



7.2 The Normal Distribution

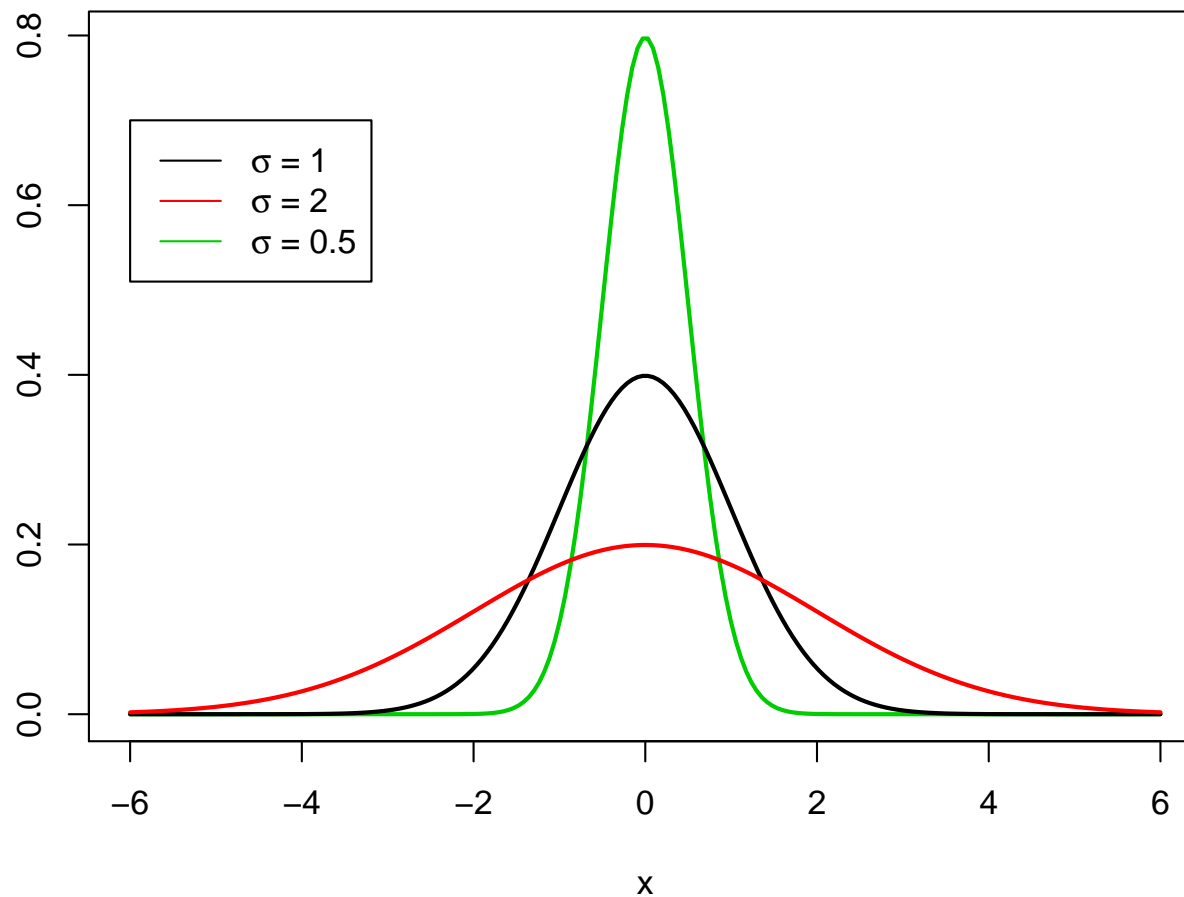
Meaning of the parameters μ and σ^2 .

- Analytical meaning of μ and σ^2 :
 - The density has its maximum at $x = \mu$.
 - The inflection points are at $x = \mu \pm \sigma$.
- Statistical meaning of μ and σ^2 : If $X \sim N(\mu, \sigma^2)$, then
 - $E(X) = \mu$,
 - $\text{var}(X) = \sigma^2$.



7.2 The Normal Distribution

This picture shows the influence of σ^2 .



7.2 The Normal Distribution

The standard normal distribution.

- If $X \sim N(0, 1)$, we say: X has a standard normal distribution.
- Let $X \sim N(0, 1)$. Using the table (and using geometry):
 - $P(X < 0) = 0.5$
 - $P(X < 1) = 0.8413$
 - $P(X > 1) = 1 - 0.8413 = 0.1587$
 - $P(-1 < X < 1) = 1 - 2 \cdot (1 - 0.8413) = 0.6826$

and:

$$P(-1.96 < X < 1.96) = 0.95$$



7.2 The Normal Distribution

The normal distribution — standardization.

- There is a table ONLY for the standard normal distribution.
- If $X \sim N(\mu, \sigma^2)$ with any $\mu \in \mathbb{R}$ and $\sigma^2 > 0$, we can standardize X :

$$Z := \frac{X - \mu}{\sigma} \sim N(0, 1)$$

- For the new random variable Z , we can again use the standard normal distribution table.



7.2 The Normal Distribution

Example 1: IQ tests.

- Let $X = \text{IQ}$ of a randomly selected adult.
- IQ tests are designed such that $X \sim N(100, 15^2)$.
- What is the probability that a randomly selected person's IQ is. . .
 - higher than 130?
 - less than 90?
 - between 90 and 110?



7.2 The Normal Distribution

Example 1: IQ tests.

- Now let

$X = \text{IQ}$ of a randomly selected student of
Bilgi University.

- Assuming $X \sim N(105, 15^2)$:
 - What is the probability that a randomly selected student's IQ is higher than 150?
 - How many of the students of Bilgi University would we expect to have an IQ above 150?



7.2 The Normal Distribution

Example 2: Viscosity.

- Viscosity measurements from a batch chemical process are assumed to be a normal random variable with mean 15 and standard deviation 1.
- What is the probability that a batch has viscosity. . .
 - above 15?
 - above 15.5?
 - above 16?



7.2 The Normal Distribution

Example 3: Body-height.

- Assume that the body-height (in cm) of a randomly selected male person is a random variable $X \sim N(178, \sigma^2)$.
- Which of the following could be a possible value of σ^2 ? —
1 / 25 / 50 / 100 / 250 / 1000 ???



7.2 The Normal Distribution

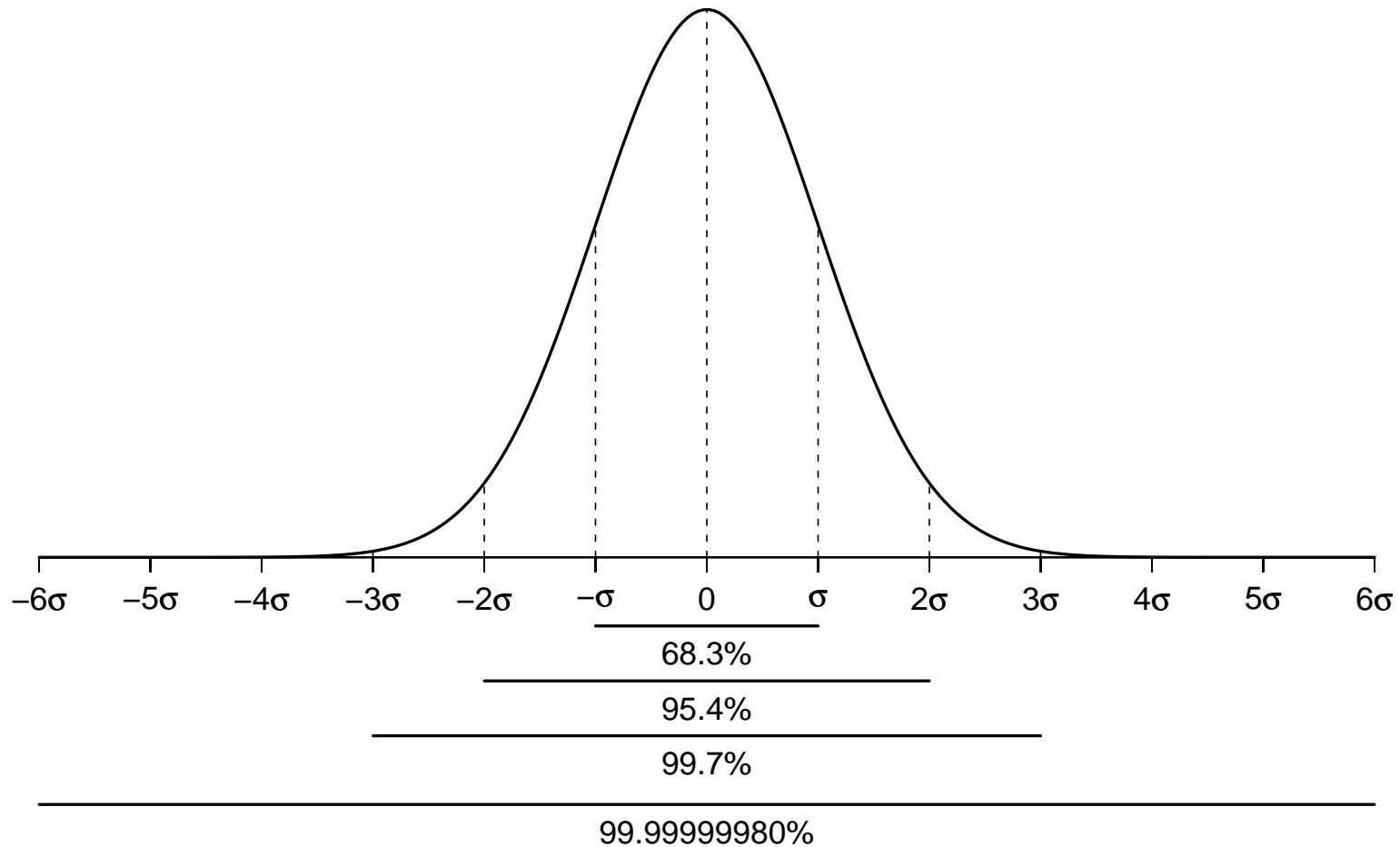
The sigma rules again.

- If $X \sim N(\mu, \sigma^2)$:
 - $P(\mu - 1 \cdot \sigma \leq X \leq \mu + 1 \cdot \sigma) \approx 0.68$
 - $P(\mu - 2 \cdot \sigma \leq X \leq \mu + 2 \cdot \sigma) \approx 0.95$
 - $P(\mu - 3 \cdot \sigma \leq X \leq \mu + 3 \cdot \sigma) \approx 0.997$
 - $P(\mu - 6 \cdot \sigma \leq X \leq \mu + 6 \cdot \sigma) \approx 0.99999999980$
- The last rule — the six- σ rule — gave rise to the six-sigma concept in quality management (Motorola, mid-1980s).



7.2 The Normal Distribution

The sigma rules again.



7.2 The Normal Distribution

Further important properties of the normal distribution.

- Let $X \sim N(\mu, \sigma^2)$. Then, for $a, b \in \mathbb{R}$:

$$aX + b \sim N(a\mu + b, a^2\sigma^2).$$

- Let $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$; X_1, X_2 independent.
Then:

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2).$$



7.2 The Normal Distribution

Further important properties of the normal distribution.

Consequences are:

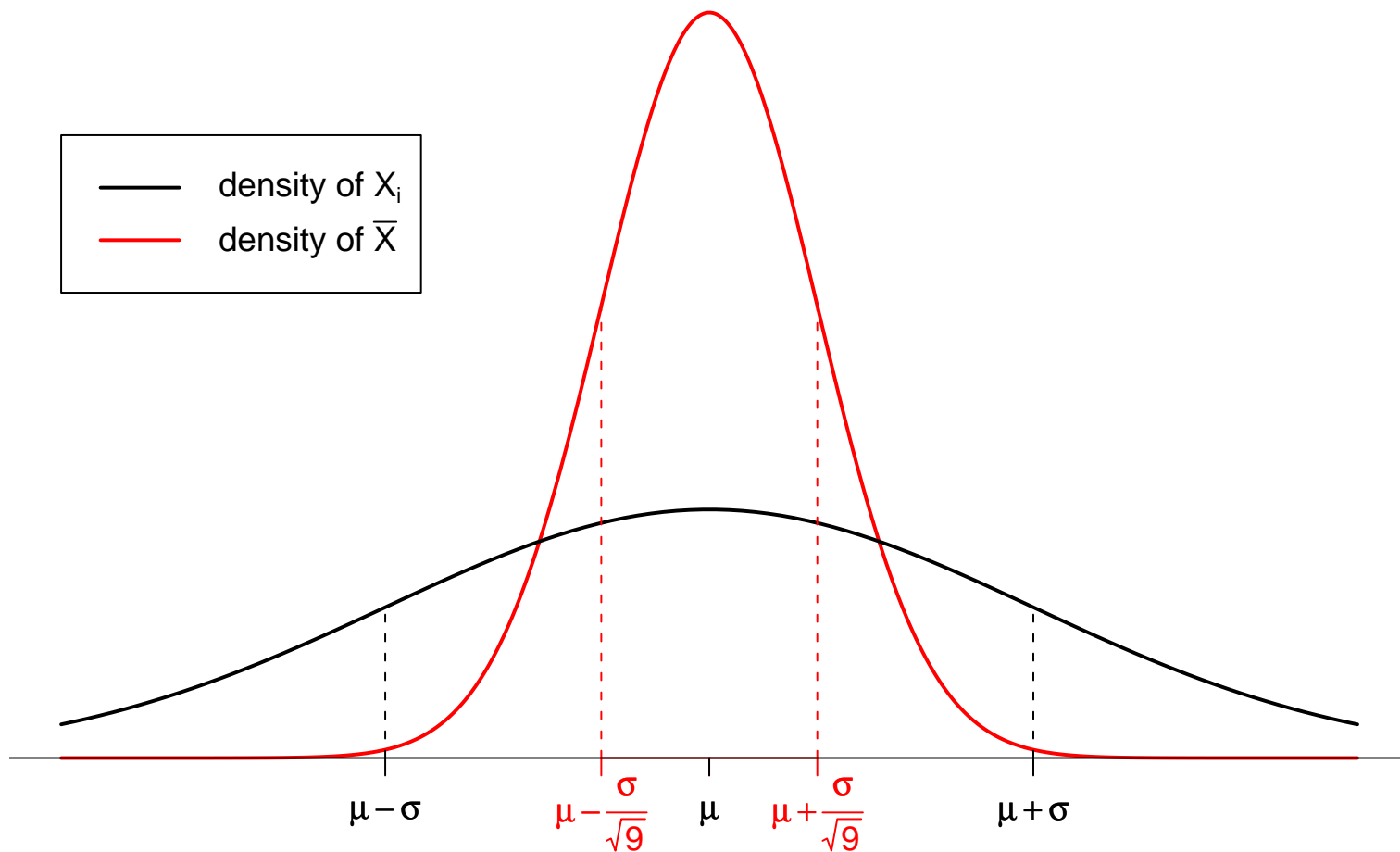
- Let $X_1, \dots, X_n \sim \text{N}(\mu, \sigma^2)$ and independent. Then:

$$\sum_{i=1}^n X_i \sim \text{N}(n\mu, n\sigma^2),$$
$$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i \sim \text{N}\left(\mu, \frac{\sigma^2}{n}\right)$$



7.2 The Normal Distribution

Densities of X_i and \bar{X} (with $n = 9$).



7.2 The Normal Distribution

Example 4: Airline passenger weight.

- An airline assumes that the weight (in kg) of a passenger, including carry-on baggage weight, is a random variable

$$X \sim N(84, 400).$$

- An airplane with 10 seats has a capacity of 1000 kg.
- Compute the probability that this limit is exceeded, i.e. that 10 passengers weigh more than 1000 kg.
- The normality assumption is not really needed. (CLT!)



7.2 The Normal Distribution

Example 5: Average IQ.

- Süleyman Hoca is offering an elective course at Bilgi University.
- Seven students have registered.
- What is the probability that the average IQ of these students is above 115?
 - Assume again that student IQ is normally distributed with mean 105 and standard deviation 15.
 - Also assume that the IQs of the students in this course are independent. (Is this assumption plausible?)



7.3 The Lognormal Distribution

Derivation of the lognormal distribution.

- If $Y \sim N(\mu, \sigma^2)$, then $X := e^Y$ is said to have a lognormal distribution with parameters μ and σ^2 .
- Its density is:

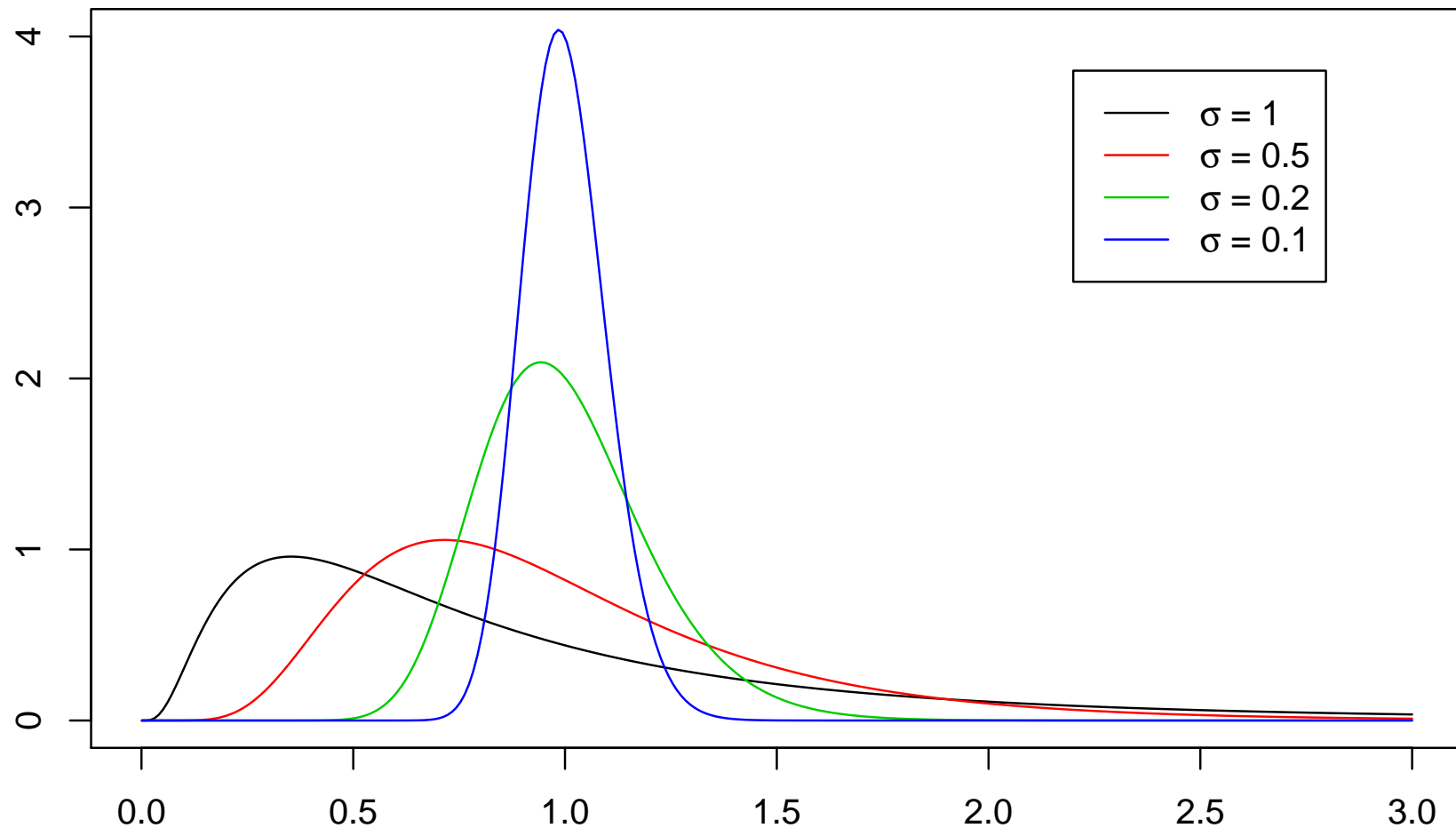
$$f(x) = \frac{1}{\sigma x \sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\sigma^2}}, \quad x > 0, \quad \mu \in \mathbb{R}, \quad \sigma > 0.$$

- In symbols: $X \sim \text{LN}(\mu, \sigma^2)$



7.3 The Lognormal Distribution

This picture shows the density. (Each expectation equals 1.)



7.3 The Lognormal Distribution

Moments of the lognormal distribution.

- Let $X \sim \text{LN}(\mu, \sigma^2)$. Then,

$$\begin{aligned} E(X) &= e^{\mu + \sigma^2/2}, \\ \text{var}(X) &= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1). \end{aligned}$$

- What is the median of X ?



7.3 The Lognormal Distribution

A financial application.

- Define stock prices:

V_0 : price today (known), V_t : price at time $t > 0$ (unknown)

- Some celebrated models (e.g., the Black-Scholes model for option pricing) assume that

$$V_t = V_0 \cdot X, \quad \text{where } X \sim \text{LN}(\mu, \sigma^2)$$

- We shall see a justification of this assumption in Chapter 8!
- Under the lognormality assumption: $E(V_t|V_0) = V_0 \cdot e^{\mu + \frac{\sigma^2}{2}}$.
(If $\mu = -\sigma^2/2$, the process (V_t) will be a martingale.)



7.3 The Lognormal Distribution

Example: Stock prices.

- Consider a certain stock, and define stock prices

V_0 (today), V_1 (one year ahead of now).

Assume: $V_1 = X \cdot V_0$, where $X \sim \text{LN}(0, 0.16)$, $V_0 = \text{€} 100$.

- With this model: What is. . .
 - the probability that the stock price is at least € 130
 - the expected stock price after one year?



7.4 The Exponential Distribution

Definition, some properties.

- A continuous random variable X is said to have an exponential distribution with parameter $\lambda > 0$ if it has the density

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

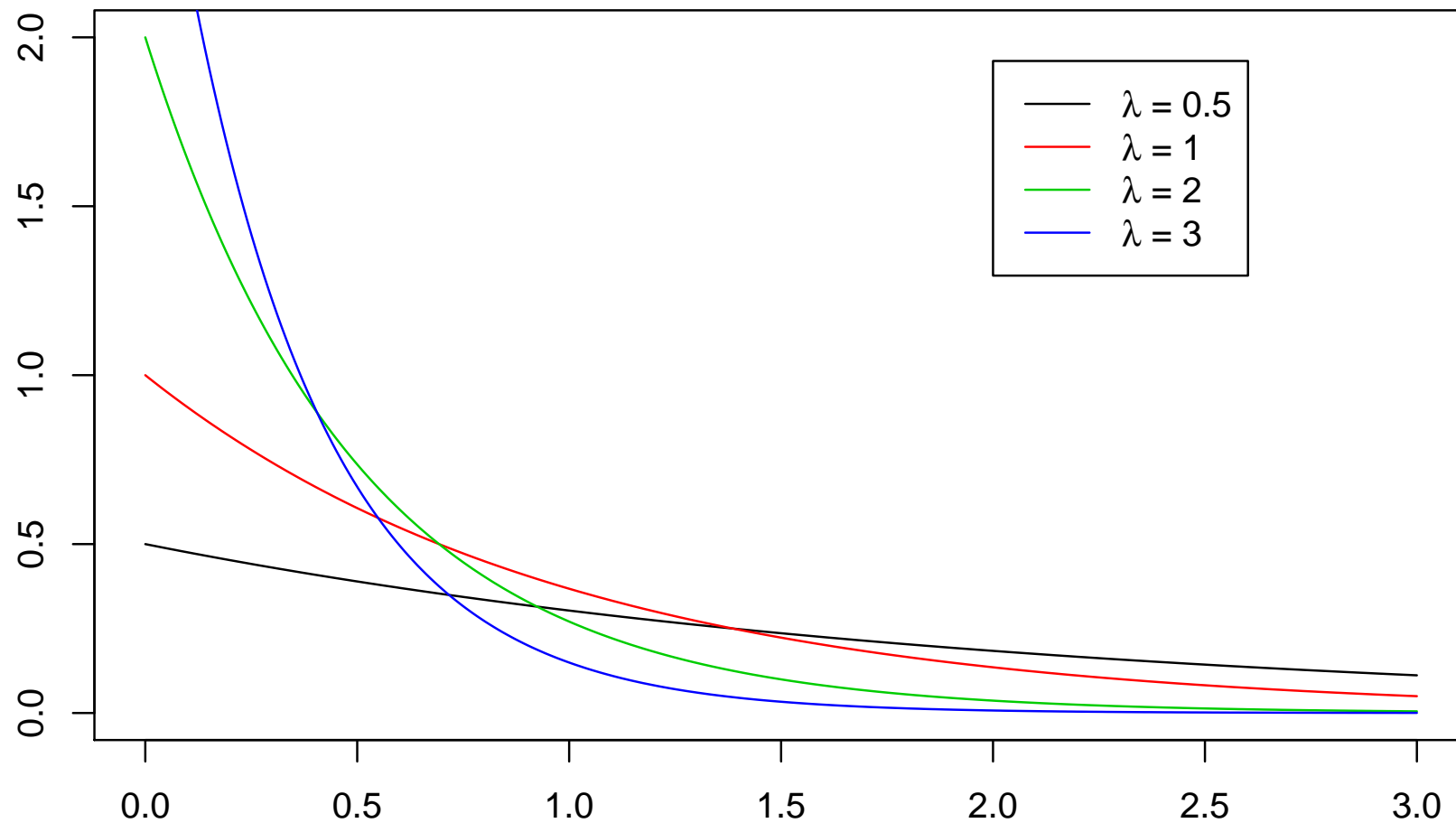
- In symbols: $X \sim \text{EXPO}(\lambda)$.
- Expectation and variance are:

$$\mathbb{E}(X) = \frac{1}{\lambda}, \quad \text{var}(X) = \frac{1}{\lambda^2}.$$



7.4 The Exponential Distribution

This picture shows the density.



7.4 The Exponential Distribution

Example: Interarrival times of customers in a copy-shop.

Interarrival times. . .

. . . before 2 p.m.:

(#)		
(11)	0*	00001122334
(6)	0●	678889
(6)	1*	033344
(2)	1●	57
(2)	2*	44
(1)	2●	7
(1)	3*	3
	3●	
	4*	
(1)	4●	5
(1)	5*	2
	5●	
<hr/>		
(31)		

. . . after 2 p.m.:

(#)		
(24)	0*	00001111112222223334444
(12)	0●	566667778899
(4)	1*	0113
(1)	1●	8
(3)	2*	002
	2●	
	3*	
(1)	3●	5
	4*	
	4●	
	5*	
	5●	
<hr/>		
(45)		

1|0=10 minutes



7.4 The Exponential Distribution

Forgetfulness.

- “Forgetfulness” or “memorylessness” of the distribution of a random variable X :

$$P(X \geq t + s | X \geq s) = P(X \geq t) \quad \text{for all } s, t > 0$$

- The following two statements are equivalent:
 - A : The distribution of the continuous random variable X is forgetful.
 - B : The random variable X is exponentially distributed.



7.4 The Exponential Distribution

The Poisson process.

- Suppose events happen according to the following assumptions:
 - The number of events occurring in $[s, s + t]$ has a Poisson distribution with parameter λt .
 - The numbers of events occurring in non-overlapping intervals are independent.
- Let $N_t = \#$ events in $[0, t]$.
- The process $(N_t)_{t \geq 0}$ is called Poisson process with intensity λ .



7.4 The Exponential Distribution

Interarrival times.

- Consider a Poisson process with intensity λ .
- Let

$X =$ length of the time interval between two successive events.

- Then, $X \sim \text{EXPO}(\lambda)$.



7.4 The Exponential Distribution

Example: Occurrence of power failures.

- Power failures occur in a certain area of Istanbul according to a Poisson process with intensity $\lambda = 0.7$ per week.
- What is the probability that there is. . .
 - a power failure during the next 12 hours?
 - no power failure during the next 3 days?
- Hint: If $X \sim \text{EXPO}(\nu)$, then the distribution function of X is

$$x \mapsto P(X \leq x) = \int_0^x \nu e^{-\nu\xi} d\xi = 1 - e^{-\nu x}.$$



7.5 The Beta Distribution

Definition, some properties.

- Its density is given by

$$f(x) = C \cdot x^{\alpha-1}(1-x)^{\beta-1}, \quad x \in (0, 1), \quad \alpha, \beta > 0.$$

- If X has this density, we write: $X \sim \text{Beta}(\alpha, \beta)$.
- If $\alpha, \beta > 1$, then:

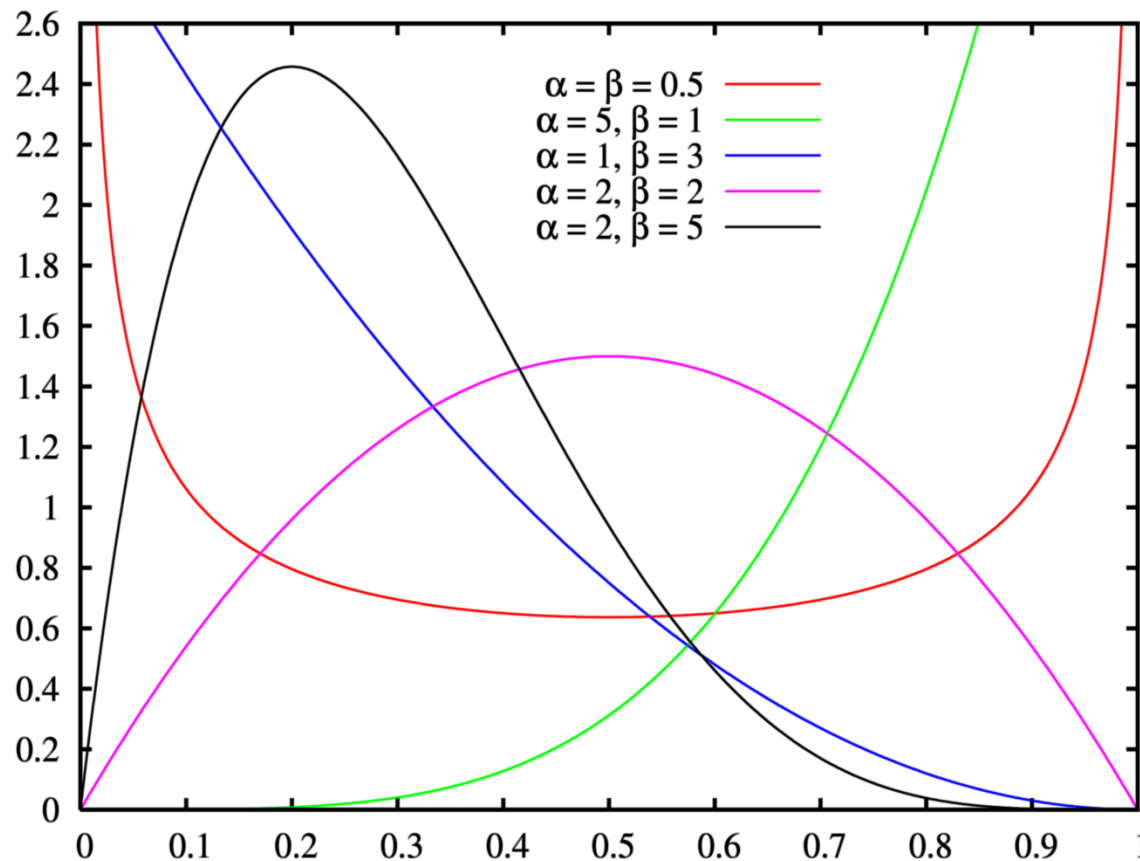
$$\mathbb{E}(X) = \frac{\alpha}{\alpha + \beta}, \quad \text{mode}(X) = \frac{\alpha - 1}{\alpha + \beta - 2},$$

$$\text{var}(X) = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)}$$



7.5 The Beta Distribution

Density of the beta distribution. (From Wikipedia.)



7.5 The Beta Distribution

Task duration modeling in PERT.

- The duration of tasks (*activity time*) is often unknown.
- Assumption: For a given task, we can specify
 - a = minimum activity time (“optimistic”)
 - b = maximum activity time (“pessimistic”)
 - m = most likely activity time (“normal”)
- Then assume that

X = duration of the task

has a beta distribution on (a, b) with $\text{mode}(X) = m$.



7.5 The Beta Distribution

The beta distribution as activity time distribution.

- Activity time results from $Y \sim \text{Beta}(\alpha, \beta)$ as

$$X = (b - a)Y + a.$$

- Problem: Values of a, b, m do not determine α, β uniquely.
- To escape this dilemma, there are approximation formulas:

$$E(X) = \frac{a + 4m + b}{6}, \quad \text{var}(X) = \left(\frac{b - a}{6}\right)^2$$



7.5 The Beta Distribution

Example: The distribution of activity times.

- We assume: minimum: $a = 5$, maximum: $b = 20$, mode: $m = 17$
- Two options to find α and β : Find α, β such that. . .
 - Option 1: . . . exact and approximate expectation values are the same.
 - Option 2: . . . exact and approximate variance values are the same.

• Numerical results:

	Option 1:		Option 2:	
	$\alpha = 4.20, \beta = 1.80$		$\alpha = 4.57, \beta = 1.89$	
	exact:	approx.:	exact:	approx.:
expectation	15.50	15.50	15.61	15.50
standard deviation	2.60	2.50	2.50	2.50

- In applications, we simply use the approximation formulas for expectation and standard deviation.



7.5 The Beta Distribution

Example: The distribution of activity times.

